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BENJAMIN F. FINKEL, Ph. D. **HERBERT E. SLAUGHT, Ph. D.**

WITH THE CO-OPERATION OF

LEONARD E. DICKSON, Ph. D.

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ON THE SYMBOLIC REPRESENTATION OF QUOTIENTIAL COEFFICIENTS OF THE SECOND ORDER.*

By ROBERT E. MORITZ, Ph. D., University of Washington.

The quotiential coefficient of a function of x, $y=f(x)$, has been defined by the expression†

$$\lim_{k=1} \log_k \frac{f(kx)}{f(x)}. \tag{1}$$

If we denote this limit by qy/qx, it may be shown that

$$\frac{qy}{qx}=\frac{x}{y}\cdot\frac{dy}{dx}, \tag{2}$$

and, for our present purpose, we may look upon (2) as the defining equation of the quotiential coefficient‡ of a function of x.

With the aid of this equation we derive immediately the following table of quotientation formulae:

*Read before the San Francisco Section of the American Mathematical Society.

†*American Journal of Mathematics*, Vol. XXIV, p. 265.

‡The term quotiential coefficient originates from the expression (1) which defines it as the limit of the logarithm of a quotient just as the differential coefficient is defined as the limit of a quotient of differences.

$$\frac{qa}{qx}=0,\ a\neq 0;\ \frac{qa}{qx}=\frac{0}{0},\ a=0; \tag{3_1}$$

$$\frac{qax^n}{qx}=n,\ \frac{qx}{qx}=1; \tag{3_2}$$

$$\frac{qa^x}{qx}=x\log a; \tag{3_3}$$

$$q\frac{\log x}{qx}=\frac{1}{\log x}. \tag{3_4}$$

If $y=u\pm v\pm w\pm$ etc., u, v, w, ..., being functions of x,

$$y\frac{qy}{qx}=u\frac{qu}{qx}\pm v\frac{qv}{qx}\pm w\frac{qw}{qx}\pm \text{ etc.} \tag{3_5}$$

If $y=u\times\div v\times\div w\times\div$ etc.,*

$$\frac{qy}{qx}=\frac{qu}{qx}\pm\frac{qv}{qx}\pm\frac{qw}{qx}\pm \text{ etc.} \tag{3_6}$$

$$\frac{qu^n}{qx}=n\frac{qu}{qx}. \tag{3_7}$$

If $u=f(z)$ and $z=\phi(x)$,

$$\frac{qu}{qx}=\frac{qu}{qz}\cdot\frac{qz}{qx}, \tag{3_8}$$

$$\frac{qu}{qx}=1/\frac{qx}{qu}. \tag{3_9}$$

If y is a function of n dependent variables, $y=F(u_1,\ u_2,\ \ldots)$, where $u_1=\phi_1(x)$, $u_2=\phi_2(x)$, etc.

$$\frac{qy}{qx}=\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}+\left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx}+\text{ etc.}, \tag{4}$$

the brackets being used in the Eulerian sense to denote partial variation of the dependent variables, *i. e.*,

$$\left(\frac{qy}{qu_1}\right)=\frac{u_1}{y}\frac{\partial y}{\partial u_1},\ \text{etc.}$$

---

*$\times\div$ is used for the multiplication sign, $\times$, with dot in each of the vertical angles.

The proof of (4) is as follows:

$$\frac{dy}{dx}=\frac{\partial y}{\partial u_1}\frac{du_1}{dx}+\frac{\partial y}{\partial u_2}\frac{du_2}{dx}+\text{ etc.},$$

hence,

$$\frac{qy}{qx}=\frac{x}{y}\frac{dy}{dx}=\frac{u_1}{y}\frac{\partial y}{\partial u_1}\cdot\frac{x}{u_1}\frac{dy}{dx}+\frac{u_2}{y}\frac{\partial y}{\partial u_2}\cdot\frac{x}{u_2}\frac{dy}{dx}+\text{ etc.}$$

$$=\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}+\left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx}+\text{ etc.}$$

gy/gx is called the total quotiential coefficient of y with respect to x, and (qy/qu_1) the partial quotiential coefficient of y with respect to x.

Of course, all the preceding formulae may also be derived without a knowledge of differentiation by applying equation (1).

A partial quotiential coefficient of a partial quotiential coefficient we call a second partial quotiential coefficient, in symbols

$$\left[\frac{q\left(\frac{qy}{qu_1}\right)}{qu_2}\right]=\frac{q^2y}{qu_2qu_1}.$$

But unlike differentiation, partial quotientiation is not commutative, so that the order of the variables must be carefully attended to. We shall follow the order observed by American writers on differentiation, namely,

$$\left(\frac{q^2y}{qu_2\ qu_1}\right)=\left[\frac{q\left(\frac{qy}{qu_1}\right)}{qu_2}\right].$$

THEOREM. *If y is a function of two variables, u and v, then*

$$\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\ qv}\right)=\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qv\ qu}\right). \tag{5}$$

Proof. By definition,

$$\left(\frac{q^2y}{qu\ qv}\right)=\frac{u}{\left(\frac{qy}{qv}\right)}\cdot\frac{\partial}{\partial u}\left(\frac{v}{y}\cdot\frac{\partial y}{\partial v}\right)=u\left(\frac{\partial^2y}{\partial u\ \partial v}-\frac{1}{y}\cdot\frac{\partial y}{\partial v}\cdot\frac{\partial y}{\partial u}\right)\Big/\frac{\partial y}{\partial v},$$

$$\left(\frac{q^2y}{qv\ qu}\right)=\frac{v}{\left(\frac{qy}{qu}\right)}\cdot\frac{\partial}{\partial v}\left(\frac{u}{y}\cdot\frac{\partial y}{\partial u}\right)=v\left(\frac{\partial^2y}{\partial v\ \partial u}-\frac{1}{y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v}\right)\Big/\frac{\partial y}{\partial u},$$

whence,

$$\left(\frac{q^2y}{qu\ qv}\right):\left(\frac{q^2y}{qv\ qu}\right)=\frac{u}{y}\cdot\frac{\partial y}{\partial u}:\frac{v}{y}\cdot\frac{\partial y}{\partial v}=\left(\frac{qy}{qu}\right):\left(\frac{qy}{qv}\right)$$

which states the theorem to be proved in the form of a proportion.

Let us next quotientiate, with respect to x, according to our rules (3_1)—(3_9), each side of the expression

$$\frac{qy}{qx}=\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}.$$

After clearing of fractions there results

$$\begin{aligned}\frac{q^2y}{qx^2}\frac{qy}{qx}=&\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)\frac{\overline{qu}^2}{qx}+\left[\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qv\ qu}\right)+\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu\ qv}\right)\right]\frac{qu}{qx}\frac{qv}{qx}\\&+\left(\frac{qy}{qv}\right)\left(\frac{q^2u}{qv^2}\right)\frac{\overline{qv}^2}{qx}+\left(\frac{qy}{qu}\right)\frac{qu}{qx}\frac{q^2u}{qx^2}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\frac{q^2v}{qx^2}.\end{aligned}\tag{6}$$

By applying (5) the expression within the brackets reduces to $2\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\ qv}\right)$ The resulting expression bears a striking resemblance to the corresponding expression for the differential quotient of the second order.

The first three terms on the right are now

$$\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)\frac{\overline{qu}^2}{qx}+2\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\ qv}\right)\frac{qu}{qx}\frac{qv}{qx}+\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right)\frac{\overline{qv}^2}{qx},$$

which, by dropping the partial quotiential coefficients of the first order and replacing at the same time q^2y by $\overline{qy}^2$, assumes the form of a perfect square, namely,

$$\left(\frac{qy}{qu}\right)^2\frac{\overline{qu}^2}{qx}+2\left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right)\frac{qu}{qx}\frac{qv}{qx}+\left(\frac{qy}{qv}\right)^2\frac{\overline{qv}^2}{qx}=\left[\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\right]^2,$$

so that (6) may be written symbolically,

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\left(\frac{qy}{qu}\right)\frac{qu}{qx}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\right]^{(2)}+\left(\frac{qy}{qu}\right)\frac{qu}{qx}\frac{q^2u}{qx^2}+\left(\frac{qy}{qv}\right)\frac{qv}{qx}\frac{q^2v}{qx^2}\tag{7}$$

where the exponent in parenthesis signifies, that the expression to which it is attached is to be squared, and after squaring, $\left(\frac{qy}{qu}\right)^2$ is to be replaced by $\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right)$, $\left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right)$ by $\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qu\ qv}\right)$, and $\left(\frac{qy}{qv}\right)^2$ by $\left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right)$.

When $u=ax^m$, $v=bx^n$, formula (7) reduces to a remarkably simple from. For in that case $\frac{qu}{qx}=m$, $\frac{qv}{qx}=n$, $\frac{q^2u}{qx^2}=0$, $\frac{q^2v}{qx^2}=0$, hence all terms on the right of (7) outside of the bracket vanish, and we have

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\left(\frac{qy}{qu}\right)m+\left(\frac{qy}{qv}\right)n\right]^{(2)}. \tag{8}$$

Formulas (6), (7), and (8) may be at once extended to functions of any number of dependent variables. Let the the dependent variables be u_1, u_2, ..., u_n, then from (4),

$$\frac{qy}{qx}=\sum_{i=1}^{i=n}\left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx}.$$

Quotientiating again and remembering that generally

$$\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_k\,qu_i}\right)=\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i\,qu_k}\right),\quad i,\ k=1,\ \dots,\ n,$$

we obtain

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\sum_{i=1}^{i=n}\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_i^{\,2}}\right)\overline{\frac{qu}{qx}}_i^{\,2}+2\sum_{\substack{k=2\\ i<k}}^{k=n}\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_i\,qu_k}\right)\frac{qu_i}{qx}\frac{qu_k}{qx}$$

$$+\sum_{i=1}^{i=r}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}, \tag{9}$$

or in symbolic representation,

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{i=n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\right]^{(2)}+\sum_{i=1}^{i=r}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}. \tag{10}$$

If, moreover, each of the functions is of the form $u_i=a_ix^{m_i}$, the second sum in expression (10) vanishes, $qu_i/qx=m_i$, and the expression reduces to

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{i=n}m_i\left(\frac{qy}{qu_i}\right)\right]^{(2)}. \tag{11}$$

SEVERAL HISTORIC PROBLEMS WHICH HAVE NOT YET BEEN SOLVED.

By DR. G. A. MILLER.

Goldbach's theorem affirms that every even number is the sum of two primes, unity being regarded as a prime number. For instance, $2=1+1$, $4=1+3=2+2$, $6=1+5=3+3$, $8=1+7=3+5$, $10=3+7=5+5$, $12=1+11=5+7$, etc. Although this theorem has been known for more than one hundred and sixty years* yet no one has succeeded in either proving or disproving it. Many attempts have been made, especially in recent years, but these have only served to make it more probable that the theorem is universally true. In two of the recent papers† an attempt is made to find some approxmate laws of increase of the number of ways in which an even number can be resolved into the sum of two primes when this even number becomes large. In the latter of these, the author reaches the conclusion that every even number which exceeds 50,000 is the sum of at least 330 different pairs of primes. It should, however, be emphasized that this result is not proved and that it still appears possible that some even number may be found which is not the sum of two primes.

Another theorem which remains unproved and has a still more extensive history than the one noted in the preceding paragraph is due to Fermat and is known under various names. Two of these are, *the last theorem of Fermat* and *the great theorem of Fermat.* It affirms that the equation

$$x^n+y^n=z^n$$

is not satisfied by any integral value of x, y, z whenever $n>2$; and it is equivalent to the theorem that the difference of the nth power of two rational fractions is never unity whenever $n>2$. The Greeks studied this equation for $n=2$ in connection with the rational right triangle. In his edition of Diophantus, Fermat states on the margin of a page that he has found a wonderful proof for the assertion that $x^n+y^n=z^n$ is not satisfied for integral values of x, y, z when $n>2$ but that the margin is too small to contain the proof. A large number of attempts have since been made but no one has been entirely successful although Kummer succeeded in proving it for an infinite number of special values of n. While it is very likely that the theorem is universally true yet it has not been *proved* impossible to find three integral values which satisfy the given equation for large values of n.

In Ball's History of Mathematics, third edition, page 306, the statement is made that it is not known whether there are any values of n to

*It is contained in letters written by Goldbach and Euler in 1742.

†Haussner, Jahresbericht der Deutschen Mathematiker-Vereinigung, Vol. 5 (1896), p. 62; Ripert, L' Intermediaire des Mathematiciens, Vol. 10 (1903), p. 78.

which Kummer's proof does not apply. This is incorrect as such values are not only known but it is also known that the number of such values increases as n increases* so that an infinite number of values of n do not come under Kummer's proof. Unfortunately Ball's History contains a large number of other inaccurate statements.†

As a third unsolved problem with a long history we may mention that no one has yet either proved or disproved the existence of an odd perfect number. A perfect number is one which like $6=1+2+3$ or $28=1+2+4+7+14$ is equal to the sum of its divisors. Euclid observed that $2^{p-1}(2^p-1)$ is a perfect number whenever 2^p-1 is a prime, and it has since been proved that every even perfect number is of this form. Nine such numbers are known. They increase so rapidly with p that their determination for large values of p is very laborious. Descartes thought it possible that odd perfect numbers exist yet none have been found although the subject has received a great deal of attention.

The questions mentioned above relate to properties of natural numbers that are easily understood. A similar problem which has a less extensive history is, whether there is an infinite number of pairs of primes such that the difference of the numbers which constitute a pair is 2. In other words, whether there is no largest prime such that this prime increased by 2 is again a prime. Many similar questions present themselves in number theory, where the very simple and the very difficult theorems seem to lie side by side without exhibiting any external evidence in regard to the class to which they belong.

A problem which has a much less extensive history than the preceding and which can probably be solved more readily is the determination of a six times transitive function which is neither symmetric nor alternating. This is equivalent to the determination of a six-fold transitive group which is neither alternating nor symmetric. In 1861, Mathien published‡ a five-fold transitive function on twelve letters and announced a similar function on twenty-four letters, which he afterwards explained more fully. Although more than forty years have passed since these discoveries no one has extended these results so that we are still ignorant of any six- or seven-fold transitive functions that are neither alternating nor symmetric. The more general question of determining a limit of transitivity for non-alternating and non-symmetric functions of a given degree has received considerable attention during this period and much progress has been made along this line.

One of the most beautiful theorems due to Lagrange affirms that every algebraic number of the second degree may be written in the form of a periodic continued fraction and that every periodic continued fraction is equal to an algebraic number of the second degree. The great importance

*Cf. Kronecker, Vorlesungen ueber Zahlentheorie, 1901, p. 23.
†Cf. On Ball's History of Mathematics, THE AMERICAN MATHEMATICAL MONTHLY, Vol. 9 (1902), p. 280.
‡Mathien, Journal de Mathematiques, Vol. 6 (1861), p. 241.

of this result led men like Jacobi and Hermite to make efforts to prove a similar theorem with respect to algebraic numbers of the third degree but their efforts were not crowned with success. Comparatively little progress has been made towards useful criteria to determine the degree of algebraic numbers, or even towards determining whether a given number is algebraic or transcendental. One of the problems suggested by Hilbert at the International Congress of Mathematicians held at Paris in 1900 is to determine whether a^p, the base being algebraic and the exponent an irrational algebraic number, always represents a transcendental or at least an irrational number.

It should not be inferred that the preceding problems are suggested as very suitable fields for the young investigator. The main object in stating them is to point out to those who may not have good library facilities that some problems relating to very elementary matters still remain unsolved, and, if possible, to encourage some one to acquaint himself with congeniel fields of study where much remains to be done. A clear understanding of the real nature of unsolved historic problems is of great importance to the stvdent since results bearing on such problems are of especial interest. Many problems of this kind are noted from time to time in *L' Intermediaire des Mathêmaticiens* published by Gauthier-Villars, and a set of about twenty very fundamental ones were given by Hilbert at the Congress mentioned in the preceding paragraph. These have been published in the *Bulletin of the American Mathematical Society,* Vol. 8 (1902), p. 437.

NOTE ON A RECENT PROBLEM IN THE AMERICAN MATHEMATICAL MONTHLY.

By R. D. CARMICHAEL, Anniston, Alabama.

The object of this note is to state in a somewhat more general form a proposition in number theory demonstrated on pages 155-156 of Volume XIII of the MONTHLY.

Given that $P^{\delta\alpha}-R^{\delta\alpha}$ is divisible by $\delta\alpha$, the necessary and sufficient conditions that the expression $\frac{P^\alpha - R^\alpha}{\delta\alpha(P-R)}$ shall be integral are: (1) *α must be divisible by e, the least integer such that P^e-R^e is divisible by α_k, where for α_k is taken in turn the various prime factors of α not dividing $P-R$;* (2) *δ is any divisor of $(P^\alpha - R^\alpha)/\alpha(P-R)$.*

The proof is practically identical with that given in the MONTHLY (*l. c.*), except in showing that δ and $I-R$ have no common factor. If δ_1 is such a prime factor, it becomes necessary in the present case to modify the proof that δ_1 must be a factor of α. This is shown as follows:

If δ_1 is contained in $P-R$, then $P=R+\delta_1 n\lambda$, where λ i some integer and $n \overline{\gtrless} 1$. Hence,

$$P^{\delta_1}=R^{\delta_1}+\delta_1{}^{n+1}\lambda+\ldots\equiv R^{\delta_1} \text{ (mod. } \delta_1{}^{n+1}).$$

Hence, $P^{\delta_1}-R^{\delta_1}$ is divisible by $\delta_1{}^{n+1}$. But $P^{\alpha}-R^{\alpha}$ is divisible by $\delta_1{}^{n+1}$. Now $P^{\delta_1}-R^{\delta_1}$ is the lowest number of this form which is divisible by $\delta_1{}^{n+1}$. Hence δ_1 is a factor of α.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

273. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

Three ingots of the precious metals were received at the Mint for assay, where it was found as follows: That in 3 grains of the first ingot and 2 grains of the second the gold was 3 times the silver; that in 2 grains of the first and 6 grains of the third the gold was 8 times the copper; that in 2 grains of the second and 3 grains of the third the silver was 5 times the copper; that in 1 grain of the first, 2 grains of the second, and 3 grains of the third the gold was 2 times the silver; that in 1 grain each of the first and second ingots there were 11 parts of gold to 5 parts of silver; and that 6 grains of the first, 5 grains of the second, and 2 grains of the third on being assayed proved to be 17 carats gold fine. There was no trace of any other metal in the ingots.

Required: The theoretical analysis of each of three ingots.

Solution by the PROPOSER.

Let x=the fraction of gold in a grain of the first ingot;
y=the fraction of silver in a grain of the first ingot;
$1-(x+y)$=the fraction of copper in a grain of the first ingot; and
z=the fraction of gold in a grain of the second ingot;
u=the fraction of silver in a grain of the second ingot;
$1-(z+u)$=the fraction of copper in a grain of the second ingot; and
v=the fraction of gold in a grain of the third ingot;
w=the fraction of silver in a grain of the third ingot;
$1-(v+w)$=the fraction of copper in a grain of the third ingot.

There are six unknown quantities and six conditions in the solution and problem which may be equated as follows:

First, $3x+2z=3[3y+2u]\ldots(1)$;
Second, $2x+6v=8\{2[1-(x+y)]+6[1-(v+w)]\}\ldots(2)$;
Third, $2u+3w=5\{2[1-(z+u)]+3[1-(v+w)]\}\ldots(3)$;
Fourth, $x+2z+3v=2(y+2u+3w)\ldots(4)$;
Fifth, $x+z : y+u :: 11 : 5\ldots(5)$; and
Sixth, $6x+5z+2v : 13 :: 17 : 24\ldots(6)$.

By elimination, we have $x=1$, $y=0$, $1-(x+y)=0$, $z=\frac{3}{8}$, $u=\frac{5}{8}$, $1-(z+u)=0$, $v=\frac{2}{3}$, $w=\frac{5}{24}$, and $1-(v+w)=\frac{1}{8}$.

We interpret these results as follows: That the first ingot was pure gold; that the second ingot was 9-carat gold, and 15-carat silver; and that the third ingot was 16-carat gold, 5-carat silver, and 3-carat copper.

Also solved in the same manner by G. B. M. Zerr, S. A. Corey, L. E. Newcomb, and G. W. Greenwood.

274. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find the limit of $\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^3-1}\cdot\frac{11^2+1}{11^2-1}\cdots$ where the squared numbers are the natural odd *primes* in order.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va., and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Putting the expression in the form

$$\frac{(1+1/3^2)(1+1/5^2)(1+1/7^2)\ldots}{(1-1/3^2)(1-1/5^2)(1-1/7^2)\ldots}=\frac{s}{s^1}$$

and remembering that $(1+1/2^2)s=\frac{15}{\pi^2}$ and $(1-1/2^2)s_1=\frac{6}{\pi^2}$, (pp. 133-134, Vol. V, No. 5), we have $s/s_1=1\frac{1}{2}$.

C. N. Schmall gives the following arithmetical solution of 269. When the boats first meet, combined distance traveled is equal to width of river; when they meet for the second time the distance traveled is equal to three times the width of river and that each boat has gone three times as far as when they first met. Hence one has gone 3×720 yards=2160 yards and has made one trip and 400 yards of the return trip· Hence, width of river =2160 yards—400 yards=1760 yards=1 mile.

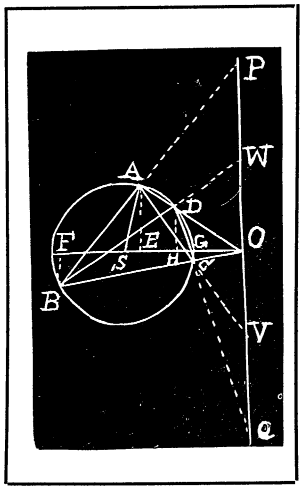
GEOMETRY.

302. Proposed by F. H. SAFFORD, Ph. D., University of Pennsylvania.

Through a given point within a circle draw any two chords, also a radius and a secant perpendicular to the radius. Let the extremities of the chords be taken as the vertices of a quadrilateral. Show that the sides of the quadrilateral, produced when necessary, cut the secant in points equidistant, in pairs, from the given point. [A proof by Euclidean geometry is preferred, as the problem was originally given to a high school class.] Must the given points be within the circle?

Solution by the PROPOSER.

The problem holds for the complete quadrilateral and for the given point taken inside or outside of the circle. Of the three pairs of intercepts on the secant, one is null, corresponding to the given point. The following proof is for the case of the given point outside of the circle and the intercepts by the *diagonals* upon the secant. The method is applicable to all cases.



In the diagram, O is the given point, A, B, C, D are the vertices of the quadrilateral, and E, F, G, H are the projections of the respective vertices upon the diameter determined by O and the center of the circle at S. Let a be the radius of the circle and let SO be b. To prove that the diagonals AC and BD, produced, cut off equal distances OV and OW on the perpendicular to SO at O. From the similar triangles OAC and OBD,

$$\frac{\triangle OAC}{\triangle OBD}=\frac{OA^2}{OB^2}=\frac{OC^2}{OD^2}\ldots(1).$$

Since $\triangle OAC=\triangle OAV-\triangle OCV$, $\triangle OBD=\triangle OBW-\triangle ODW$, and using OV and OW as bases, it follows that

$$\frac{\triangle OAC}{\triangle OBD}=\frac{OV(OE-OG)}{OW(OF-OH)}\ldots(2).$$

From (1) and (2),

$$\frac{OV}{OW}=\frac{OA^2}{OB^2}\cdot\frac{OF-OH}{OE-OG}\ldots(3).$$

Noticing that OE and OG are the projections of OA and OC, the triangles SOA and SOC give, $a^2=b^2+OA^2-2b.OE$, $a^2=b^2+OC^2-2b.OG$.

$$\left.\begin{aligned}\therefore OE-OG&=\frac{OA^2-OC^2}{2b}\\ \text{Similarly, } OF-OH&=\frac{OB^2-OD^2}{2b}\end{aligned}\right\}\ldots(4).$$

From (3) and (4), using also the latter part of (1),

$$\frac{OV}{OW}=\frac{OA^2}{OB^2}\cdot\frac{OB^2-OD^2}{OA^2-OC^2}=\frac{OA^2.OB^2-OA^2.OD^2}{OB^2.OA^2-OB^2.OC^2}=1.$$

Also solved by C. N. Schmall, A H. Holmes, and L. E. Newcomb. Mr. Newcomb's demonstration was exhaustive, covering the cases when the point is within and without the circle. Our space is too limited to publish his demonstration.

CALCULUS.

230. Proposed by C. N. SCHMALL, College of the City of New York.

The greatest rectangle is inscribed in an ellipse, and the greatest ellipse in that rectangle, again the greatest rectangle in that (second) ellipse, and the greatest ellipse in that (second) rectangle, and so on *ad infinitum;* show that the sum of all the inscribed rectangles is equal to the area of the rectangle circumscribed about the given ellipse.

Solution by J. E. SANDERS, Reinersville, Ohio, and A. H. HOLMES, Brunswick, Maine.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, be the equation of the ellipse. Let $2x_1$ be the length of its maximum inscribed rectangle. Then $2y_1=2\sqrt{(a^2b^2-b^2x_1^2)}=$its width, and the area of the rectangle is $4x_1y_1=4bx_1\sqrt{(a^2-x_1^2)}/a=$maximum.

$\therefore 4b\sqrt{(a_2-x_1^2)}-\frac{4bx^2}{\sqrt{(a^2-x_1^2)}}=0$; from which $x_1=\frac{1}{2}\sqrt{2}\,a=a_1$. Hence, $y_1=\frac{1}{2}\sqrt{2}\,b=b_1$. Hence its area$=4a_1b_1=2ab$. The equation of the ellipse inscribed in this rectangle is $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$. Hence, the length of the maximum rectangle inscribed in this ellipse is $2(\frac{1}{2}\sqrt{2}\,a_1)=2a_2$, its width is $2(\frac{1}{2}\sqrt{2}\,b_1)=2b_2$, and its area is $4a_2b_2=2a_1b_1=ab$. By induction, the area of the nth rectangle$=\frac{1}{2}$ of the area of the $(n-1)$th rectangle, or $a_nb_n=\frac{1}{2}a_{n-1}b_{n-1}$. Hence, the sum of all the rectangles is $S=2ab+ab+\frac{1}{2}ab+\frac{1}{4}ab+\frac{1}{8}ab+\ldots$ *ad infinitum*$=4ab$, which is the area of the rectangle circumscribing the original ellipse.

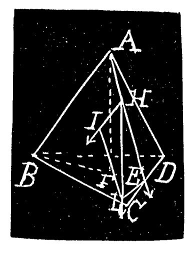
Also solved by G. B. M. Zerr, J. Scheffer, and G. W. Greenwood.

MECHANICS.

193. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three light smoothly jointed rods stand like a tripod—the three edges of a regular tetrahedron. A rectangular board, weight w, stands on this like an easel. Find the thrust on the rod which does not touch the easel.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.



Let $AB=a$ be one of the equal rods jointed at A; H the center of gravity of the board, weight w; HL the direction of the weight w. Resolve HL into the components HE along the median AE, and HI parallel to AB. Then HI is the thrust required. Now $AE=BE=\frac{1}{2}a\sqrt{3}$. Draw AF perpendicular to BE. Then $BF=\frac{2}{3}BE=\frac{1}{3}a\sqrt{3}$. $EF=\frac{1}{6}a\sqrt{3}$.

$\cos BAE=\cos IHE=\frac{1}{3}\sqrt{3}$, $\sin IHE=\frac{1}{3}\sqrt{6}$.

$\sin IHL = \frac{1}{3}\sqrt{3}$, $\cos IHL = \frac{1}{3}\sqrt{6}$.

$\sin EHL = \frac{1}{3}$, $\cos EHL = \frac{2}{3}\sqrt{2}$.

$HI : HL = \sin ILH : \sin HIL = \sin EHL : \sin IHE$. $\therefore HI : w = \frac{1}{3} : \frac{1}{3}\sqrt{6}$ or $HI = w/\sqrt{6} = w\sqrt{6}/6$; $HE : w = \sin IHL : \sin HIL$; $HE : w = \frac{1}{3}\sqrt{3} : \frac{1}{3}\sqrt{6}$, or $HE = w/\sqrt{2} = w\sqrt{2}/2$.

Also solved by G. W. Greenwood.

AVERAGE AND PROBABILITY.

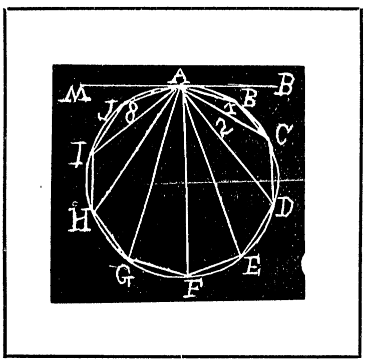
163. **Proposed by R. D. CARMICHAEL, Anniston, Ala.**

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle?

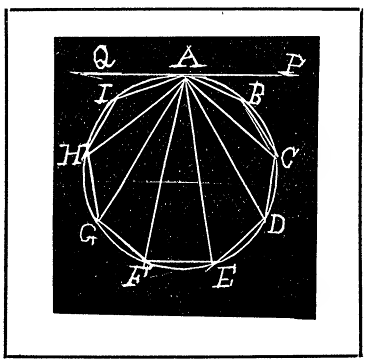
Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Whether n be even or odd, there can be formed, at any vertex, as B, or C, or D, etc., by joining it with A and any other vertex, $(n-2)$ triangles. Since there are n vertices, the total number of triangles $= n(n-2)$.

I. n even. For A the sum of the areas is the area of the polygon $= \frac{1}{2}nr^2\sin(2\pi/n)$; for B and J combined $nr^2\sin(2\pi/n) =$ sum of areas; for C and I, the sum is $2nr^2\sin(2\pi/n) - 4(n-2)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$; for D and H, $3nr^2\sin(2\pi/n) - 4nr^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n) - 4(n-4)r^2\sin(\pi/n) \times \sin(2\pi/n)\sin(3\pi/n)$. For the next pair of vertices the sum is



n even.



n odd.

$$4n^2\sin(2\pi/n) - 4(n+2)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-4(n-2)r^2\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)$$
$$-4(n-6)r^2\sin(\pi/n)\sin(3\pi/n)\sin(4\pi/n).$$

The sum of the areas of the triangles for $\frac{n-2}{2}$th, the $\frac{n}{2}$th and the $\frac{n+2}{2}$th vertices are equal, and their combined sum for the three vertices is

$$\frac{3(n-2)}{4}r^2\sin(2\pi/n)-6(2n-8)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-6(2n-12)r^2\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)$$
$$-6(2n-16)r^2\sin(\pi/n)\sin(3\pi/n)\sin(4\pi/n)$$
$$-\dots-6(4)\,r^2\sin(\pi/n)\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi.$$

The total sum for all is

$$[\tfrac{1}{2}n+n(1+2+3+\dots+\frac{n-2}{2})\ +\frac{n(n-2)}{4}]r^2\sin(2\pi/n)$$
$$-[4r^2(n-2+n+n+2+n+4+\dots+2n-8)$$
$$+2r^2(2n-8)]\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-[4r^2(n-4+n-2+n+n+2+\dots+2n-12)$$
$$+2r^2(2n-12)]\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)-\text{etc.},$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-3r^2\sin(\pi/n)\,[(n-2)(n-4)\sin(\pi/n)\sin(2\pi/n)$$
$$+(n-4)(n-6)\sin(2\pi/n)\sin(3\pi/n)+(n-6)(n-8)\sin(3\pi/n)\sin(4\pi/n)+\dots$$
$$+8\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi]$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-24r^2\,[\cos(\pi/n)\cos(2\pi/n)+3\cos(2\pi/n)\cos(3\pi/n)$$
$$+6\cos(3\pi/n)\cos(4\pi/n)+10\cos(4\pi/n)\cos(5\pi/n)+\dots$$
$$+\frac{(n-2)(n-4)}{8}\ \cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi]\sin(\pi/n)$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-12r^2\sin(\pi/n)\cos(\pi/n)\,(1+3+6+10+\dots+\frac{(n-2)(n-4)}{8}]$$
$$-12r^2\sin(\pi/n)\,[\cos(3\pi/n)+3\cos(5\pi/n)+6\cos(7\pi/n)+\dots$$
$$+\frac{(n-2)(n-4)}{8}\ \cos\frac{n-3}{n}\pi]=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-\tfrac{1}{8}n(n-2)(n-4)r^2\sin(2\pi/n)$$
$$+\frac{3n[(n-2)\cos(2\pi/n)-(n-4)]r^2\sin(2\pi/n)}{8[\sin(\pi/n)]^2}$$
$$=\frac{r^2n\{(3n-2)[\sin(\pi/n)]^2-1\}\cos(\pi/n)}{2\sin(\pi/n)}$$

$\therefore$ Average area is $\dfrac{r^2\cos(\pi/n)}{2(n-2)\sin(\pi/n)}\{[3n-2][\sin(\pi/n)]^2-1\}.$

II. n odd. By a similar process we easily get for the total sum

$$[\tfrac{1}{2}nr^2+nr^2(1+2+3+4+\ldots+\frac{n-1}{2})]\sin(2\pi/n)$$

$$-4r^2(n-2+n+n+2+n+4+\ldots+2n-7)\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$

$$-4r^2(n-4+n-2+n+n+2+\ldots+2n-11)\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)\ -\text{ etc.},$$

$$=\frac{n^3+3n}{8}r^2\sin(2\pi/n)-3r^2\sin(\pi/n)[(n-1)(n-3)\sin(\pi/n)\sin(2\pi/n)$$

$$+(n-3)(n-5)\sin(2\pi/n)\sin(3n/\pi)+(n-5)(n-7)\sin(3\pi/n)\sin(5\pi/n)+\ldots$$

$$+8\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi]$$

$$=\frac{n^3+3n}{8}r^2\sin(2\pi/n)-24r^2\sin(\pi/n)[\cos(\pi/2n)\cos(3\pi/n)$$

$$+3\cos(3\pi/n)\cos(5\pi/n)+6\cos(5\pi/n)\cos(7\pi/n)+\ldots$$

$$+\frac{(n-1)(n-3)}{8}\cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi]=\frac{n^3+3n}{8}r^2\sin(2\pi/n)$$

$$-12r^2\sin(\pi/n)\cos(\pi/n)[1+3+6+10+\ldots+\frac{(n-1)(n-3)}{8}]$$

$$-12r^2\sin(\pi/n)[\cos(2\pi/n)+3\cos(4\pi/n)+6\cos(6\pi/n+\ldots$$

$$+\frac{(n-1)(n-3)}{8}\cos\frac{(n-3)\pi}{n}]=\frac{n^3+3n}{8}r^2\sin(2\pi/n)$$

$$-\tfrac{1}{8}(n+1)(n-1)(n-3)r^2\sin(2\pi/n)+\frac{3r^2}{2\sin(\pi/n)}$$

$$-\frac{[3(n^2-2n-3)\sin(2\pi/n)-3(n^2-1)\sin(2\pi/n)\cos(2\pi/n)]r^2}{8[\sin(\pi/n)]^2}$$

$$=\frac{r^2}{4\sin(\pi/n)}\{(4n+3-3n^2)[\sin(\pi/n)]^2\cos(\pi/n)+6+6(n+1)\cos(\pi/n)\}.$$

The average area is

$$\frac{r^2}{4n(n-2)\sin(\pi/n)}\{6+6(n+1)\cos(\pi/n)-(3n^2-4n-3)\cos(\pi/n)[\sin(\pi/n)]^2\}.$$

For the foregoing solutions the following explanation is necessary.

$$2\sin\frac{(p-q)\pi}{n}\sin\frac{(p-q+1)\pi}{n}=\cos(\pi/n)+\cos\frac{(2p-2q+1)\pi}{n}$$

$$2\cos\frac{(p-q)\pi}{2n}\cos\frac{(p-q+2)\pi}{2\pi}=\cos(\pi/n)+\cos\frac{(p-q+1)\pi}{n}$$

Let $C=\cos3\theta+3\cos5\theta+6\cos7\theta+\ldots+\frac{(n-2)(n-4)}{8}\cos(n-3)\theta,$

$$S = i\sin 3\theta + 3i\sin 5\theta + 6i\sin 7\theta + \ldots + \frac{(n-2)(n-4)}{8} i\sin(n-3)\theta,$$

where $\theta = \pi/n$, $i = \sqrt{-1}$.

$$\therefore C+S = \cos 3\theta + i\sin 3\theta + 3(\cos 5\theta + i\sin 5\theta) + 6(\cos 7\theta + i\sin 7\theta) + \ldots$$

$$+\frac{(n-2)(n-4)}{8}[\cos(n-3)\theta + i\sin(n-3)\theta] = (\cos\theta + i\sin\theta)^3 + 3(\cos\theta + i\sin\theta)^5$$

$$+6(\cos\theta + i\sin\theta)^7 + \ldots + \frac{(n-2)(n-4)}{8}(\cos\theta + i\sin\theta)^{n-3} = e^{3i\theta} + 3e^{5i\theta} + 6e^{7i\theta} + \ldots$$

$$+\frac{(n-2)(n-4)}{8}e^{(n-3)i\theta} = y^3[1 + 3y^2 + 6y^4 + 10y^6 + \ldots + \frac{(n-2)(n-4)}{8}y^{n-6}]$$

$$= \frac{y^3}{(1-y^2)^3} - \frac{(n-)(n-4)y^{n+3} + n(n-2)y^{n-1} - 2n(n-4)y^{n+1}}{8(1-y^2)^3}$$

Putting $y = e^{i\theta} = \cos\theta + i\sin\theta$, and equating C=the rational part, we get

$$C = \frac{2n(n-4)\sin(2\pi/n) - n(n-2)\sin(4\pi/n)}{64[\sin(\pi/n)]^3}.$$

MISCELLANEOUS.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the number of real roots of the equation $100\sin x = x$, and show the largest root is approximately 96.10. Find $\tan 39°$ to three places of decimals. How many real roots of $\tan x = 1/x^2$ lie between 0 and 2π?

Solution by HENRY HEATON, Belfield, N. D.

(1). The equation may be written $x/\sin x = 100$, As x varies from 0 to $\frac{1}{2}\pi$, $x/\sin x$ takes the successive values between 1 and $\frac{1}{2}\pi$. Hence the equation has no real root in the first quadrant. As x varies from $\frac{1}{2}\pi$ to π, $x/\sin x$ takes all the successive values from $\frac{1}{2}\pi$ to ∞. Hence the equation has one real root in the second quadrant. In the third and fourth quadrants $x/\sin x$ is negative. Hence there can be no real root. At every round as x passes through the first and second quadrant $x/\sin x$ varies from ∞ to a minimum value then back to ∞. This minimum value is in the first quadrant when $x = \tan x$. After the first round as long as $x < 100$ there are two real positive roots to every round. Hence there can be no real positive root when $x > 31\pi$, and the whole number of real positive roots is 31. The number of real negotive roots is evidently the same with the same numerical values. Hence the whole number of real roots is 62. The largest real root evidently occurs between $x = 30\frac{1}{2}\pi$ and 31π. Put $x = y + 30\frac{1}{2}\pi$. Then $\sin(y + 30\frac{1}{2}\pi) = \frac{1}{100}(y + 30\frac{1}{2}\pi)$.

$\therefore \cos y = .958185 + \frac{1}{100}y$.

If $\cos y = .95818$, $y = 16° \ 38' = .2903^{(r)}$, a first approximate value of y. If $\cos y = .95818 + .002903 = .96108$, $y = 16° \ 2' = .27983^{(r)}$, a second approximation.

If $\cos y = .95818 + .002798 = .96098$, $y = 16^\circ\ 3\frac{1}{2}' = .28027^{(r)}$, a third approximation. If $\cos y = .95818 + .00281 = .96099$, $y = 16^\circ\ 3\frac{3}{8}' = .28023^{(r)}$, a fourth approximation. Whence $x = 96.098$, nearly.

(2). From the well known length of the side of the inscribed decagon we have $\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5}-1)$.

$$\text{Hence } \cos 36^\circ = \sqrt{\frac{1+\frac{1}{4}(\sqrt{5}-1)}{2}} = \tfrac{1}{4}\sqrt{(6+2\sqrt{5})},$$

$$\text{and } \sin 36^\circ = \sqrt{\frac{1-\frac{1}{4}(\sqrt{5}-1)}{2}} = \tfrac{1}{4}\sqrt{(10-2\sqrt{5})}.$$

$$\text{Hence } \cot 36^\circ = \tan 54^\circ = \sqrt{\frac{6+2\sqrt{5}}{10-2\sqrt{5}}} = \sqrt{(1+\tfrac{2}{5}\sqrt{5})},$$

$$\tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = 2-\sqrt{3}.$$

$$\therefore \tan 39^\circ = \tan(54^\circ - 15^\circ) = \frac{\sqrt{(1+\frac{2}{5}\sqrt{5})}-2+\sqrt{3}}{1+(2-\sqrt{3})\sqrt{(1+\frac{2}{5}\sqrt{5})}}$$

$$= \frac{\sqrt{(5+2\sqrt{5})}-2\sqrt{5}+\sqrt{15}}{\sqrt{5}+2\sqrt{(5+\sqrt{5})}-\sqrt{(15+6\sqrt{5})}}.$$

This can be readily computed to any desired degree of accuracy.

(3). As x varies from 0 to $\frac{1}{2}\pi$, $x^2\tan x$ passes through all successive values from 0 to ∞. Hence, it has one value at which $x^2\tan x = 1$. The same is true as x varies from π to $3\pi/2$. But in the second and fourth quadrants all values of $x^2\tan x$ are negative. Hence the equation has no real roots in those quadrants. Hence the number of real roots between 0 and 2π is 2.

Also solved by G. B. M. Zerr and A. H. Holmes. Dr. Zerr finds $\tan 39^\circ = .809785$ by means of $\tan 30^\circ$ and $\tan 9^\circ$. In (3), he finds $x = 51^\circ\ 17\frac{1}{2}'$ and $185^\circ\ 27'\ 10''$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

279. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

The United States Panama Canal Bonds were issued, to date August 1, 1906, and will mature on August 1, 1936; and they bear interest at the rate of 2% per annum, payable quarterly, on the first day of November, 1906, and the first day of February, May, and August, 1907, and so on for each succeeding quarter, until the bonds mature, when the principal will be paid at par with the last quarter's interest. The coupon bonds of this loan were quoted on the New York Stock Exchange, at 10.30 a. m., on December 17, 1906, at $103\frac{3}{4}$ bid and $104\frac{3}{4}$ asked.

Required: The rate of interest per annum, payable quarterly, an investor would *realize* if he purchased the Panama bonds on December 17, 1906, and could reinvest his interest income, quarterly, at the *realized* rate.

GEOMETRY.

311. **Proposed by J. OWEN MAHONEY, B. E., M. Sc., Dallas High School, Dallas, Texas.**

Triangle ABC is obtuse-angled at C; x, y, z are squares on the sides AC, CB, BA; MN and QR are lines joining adjacent sides of x, z and y, z. The common chord of the circles on MN and QR as diameters passes through C and the mid-point of NR.

312. **Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania, Philadelphia, Pa.**

A variable circle passes through a fixed point and is tangent to a given circle. If a diameter of the first circle passes through the fixed point find the locus of its other extremity.

CALCULUS.

235. **Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.**

The latitude of a place and two circles parallel to the horizon being given, to determine the declination of a heavenly body whose apparent time of passage from one circle to the other shall be a minimum.

236. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

Solve the partial differential equation, $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial u}{\partial x}$.

MECHANICS.

199. **Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.**

A sphere of water, radius $\frac{1}{49}$, the earth's radius, is brought together by mutual attractions of particles from a state of infinite diffusion. Find its temperature owing to the amount of work done by these forces.

AVERAGE AND PROBABILITY.

186. **Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.**

An urn contains $n=100$ balls; $a=25$ balls are stamped, at random, with the letter A; $b=30$ balls are stamped, at random, with the letter B; $c=40$ balls are stamped, at random, with the letter C; $d=50$ balls are stamped, at random, with the letter D. One ball is drawn at random; find the chance it has on it no letter, the letter A, or B, or C, or D, or the letters AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, or $ABCD$.

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ON A SET OF FOUR LINEAR ASSOCIATIVE ALGEBRAIC UNITS.

By S. A. COREY, Hiteman, Iowa.

Dr. Dickson has called attention in the November MONTHLY to a very remarkable set of non-associative algebraic units, and notes that among the many other similar (though usually associative) units, the best known are Hamilton's famous unit vectors. i, j, k, which on account of being perfectly isotropic when applied to Euclidean space, have been of much use to scientists in simplifying the language of analysis as applied to the physical sciences. In this connection it may be of interest to notice very briefly another somewhat remarkable set of "units," in terms of which, when interpreted in a certain manner, scalars, vectors, and quaternions may all alike be written.

We shall begin, as we may, by defining these "units" by a "multiplication table." This is virtually what Hamilton did when he defined his "units" in such a way as to admit of their bearing the interpretation he desired to place on them as unit vectors. Let, then, H, I, J, K, be these four "units" of which the following is the "multiplication table:"

	H	I	J	K
H	H	I	H	$-I$
I	H	I	$-H$	I
J	J	$-K$	J	K
K	$-J$	K	J	K

Further assume that multiplication of these "units" with ± 1 is always commutative, *e. g.* $H(-J)=-HJ$.

Then will multiplication always be associative; for, taking these "units" in sets of three, thus,

$$H.IJ=-HH=-H, \quad HI.J=IJ=-H, \quad J.HK=-JI=K, \quad JH.K=JK=K,$$

and so on until all possible permutations of the four "units" taken three at

a time have been enumerated. In every case multiplication will be found to be associative. Similarly taking the four "units" four at a time, thus,

$$HIJK=H.IJK=HIJ.K=H.I.JK=H.IJ.K=HI.JK=IK=I,$$
$$HJIK=H.JIK=HJI.K=H.J.IK=H.JI.K=HJ.IK=HI=I,$$

and so on, until all possible permutations of the four "units" taken four at a time have been enumerated. In every case multiplication will be found to be associative. But as any and all permutations or combinations of the four "units" taken five or more at a time can readily be reduced to some of the foregoing permutations of four letters, it follows that multiplication is *always* associative.

After thus defining our "units" we may impose further restrictions, provided only these restrictions in no way conflict with the restrictions imposed by our "multiplication table." By so doing we simply exclude from consideration all interpretations of these "units" which do not admit of these added restrictions. Let these added restrictions be as follows:

First. Our "units" may be added (and subtracted).

Second. Addition (and subtraction) must be associative, $\{[H+I)+J+K]=[H+(I+J)+K]$, and so on$\}$.

Third. Addition (and subtraction) must be commutative, $[(H+I+J+K)=(J+H+K+I)$, and so on$]$.

Fourth. Multiplication must be distributive over addition, $[H(H+I+K+J)=(HH+HI+HK+HJ)$, and so on$]$.

Fifth. Multiplication, addition (and subtraction) of our "units" with ordinary scalars (rational and irrational) must be associative, commutative, and distributive, exactly as if these "units" were ordinary scalars.

It should be particularly noticed that no assumption has been made that division is always possible, nor that the law of indices holds. For this reason no operation on our "units" involving either division or the law of indices is permissible, until, in any particular case, these operations have been proven permissible.

We may now seek to interpret the meaning of these "units" in such a way as to satisfy all the imposed conditions or restrictions. With this purpose in view it may be noted that all the restrictions imposed, except those imposed by the "multiplication table" are the same as those that apply to algebraic numbers (scalars) and vectors. We know that vector multiplication is non-commutative. In view of these facts it may be worth while to try to so combine algebraic numbers and vectors as to form combinations the multiplication of which is in accordance with the above "multiplication table." For, should we succeed in forming such combinations, they must fulfill *all* the requirements imposed on our "units," and, therefore, may be treated as *particular* values of these "units." No great difficulty is involved in finding a number of combinations which do not change in value when

squared (none of our "units" change in value when squared), and but little ingenuity is required to combine a set of four of them in such a way as to fulfill all the stated requirements. One such set is

$$\left.\begin{aligned}
H&=\tfrac{1}{2}[1-j-\theta(i+k)]\\
I&=\tfrac{1}{2}[1+j+\theta(i-k)]\\
J&=\tfrac{1}{2}[1+j-\theta(i-k)]\\
K&=\tfrac{1}{2}[1-j+\theta(i+k)] \text{ whence}\\
1&=\tfrac{1}{2}(H+I+J+K)\\
i&=\frac{\theta}{2}(H-I+J-K)\\
j&=\tfrac{1}{2}(-H+I+J-K)\\
k&=\frac{\theta}{2}(H+I-J-K)
\end{aligned}\right\} \quad (A)$$

where $\theta=\sqrt{-1}$, and i, j, ank k are Hamilton's unit vectors. Other similar combinations can readily be obtained from this set by a cyclical permutation of the unit vectors, i, j, k, or by substituting for these vectors other rectangular unit vectors. All such combinations must, however, be considered as mere variations of (A). We know that in ordinary algebra we may substitute such operators as d/dx, d/dy, d/dz, etc., for ordinary algebraic numbers because their laws of combination are identical. Whether, likewise, such distinct values of H, I, J, and K exist is a matter of speculation, but seems not impossible.

It might have been a more logical and simple process to have commenced by assuming (A) and then deducing the "multiplication table" therefrom. Such a method of procedure would have led up to the same results, but would not have made it so evident that the values of the fundamental "units" given in (A) are mere *particular* values.

By closely examining (A) we learn that this interpretation of our "units" implies the following:

First. Each "unit" is a Cayleyan nullitat (quaternion with zero norm).

Second. All quaternions, including scalars and vectors, can be written in terms of H, I, J, and K, and vice versa.

Third. These "units" may be algebraically combined in accordance with the same laws that govern the combination of quaternions. (If division and the law of indices be excepted, there seems to be no reason why the laws of quaternions should not govern in the combination of other entirely distinct interpretations of these "units," should such interpretations be found.)

Let $Q=(aH+bI+cJ+dK)$, $R=(eH+fI+gJ+hK)$, $W=(rH+sI+tJ+uK)$, a, b, c, d, e, f, g, h, r, s, t, and u, being numbers in the algebraic

field, operators, such as d/dx, d/dy, d/dz, etc., or a combination of such numbers and operators. Then, if $Q.R=W$,

$$r=e(a+c)+f(a-c),\ s=e(b-d)+f(b+d),$$
$$t=g(c+a)+h(c-a),\ \text{and}\ u=g(d-b)+h(d+b).$$

Using ordinary quaternion methods and Hamilton's notation,

$$SQ=\tfrac{1}{2}(a+b+c+d),$$
$$VQ=(Q-SQ)=(a-w)H+(b-w)I+(c-w)J+(d-w)K,$$
$$[w\equiv\tfrac{1}{4}(a+b+c+d)],$$
$$TQ=\sqrt{[2(ad+bc)]},\ \text{and}$$
$$UQ=(2ad+2bc)^{-\frac{1}{2}}(aH+bI+cJ+dK).$$

Similar expressions can readily be found for SUQ, VUQ, TVQ, $TVUQ$, etc. It would, indeed, be possible to construct an entire system of quaternions in which H, I, J, K would be the "units" used to replace unity and the i, j, k of Hamilton. Such a system would necessitate the use of irrational numbers in expressions involving ordinary vectors, and would, therefore, not be well suited for use in problems which concern the physicist; it would, however, have the remarkable characteristic that scalars, vectors, and vector products could all alike be expressed in terms of the same linear, homogeneous units, although such expressions would be cumbersome and of little practical value to the physicist. It is a matter of historic interest that Hamilton was much concerned about the heterogeneous character of his vector products, and tried to find an "*extra-spacial*" *unit* which would render such products homogeneous.* The $\sqrt{-1}$ of algebra may, perhaps, not be considered such an "*extra-spacial*" *unit*, but may evidently be used to obtain the result Hamilton sought to obtain by the use of such a unit. Inasmuch as the heterogeneity of vector multiplication disappears when $\sqrt{-1}$ is introduced by the linear substitutions involved in these "units," we may, perhaps, better conclude that the heterogeneity which Hamilton sought to remove was, after all, only a seeming heterogeneity resulting from the point of view afforded by his particular system of fundamental unit vectors, whereas quaternion analysis as a system is, of course, independent of any particular set of fundamental unit vectors.

But, in conclusion, we may observe that the algebraic properties of these "units" had been defined and proven consistent before any attempt was made to give them an interpretation. Had we, indeed, entirely failed to find an interpretation, these "units" would, nevertheless, have been realities, in an abstract sense, to the pure algebraist. The fact that they happen to be isomorphic to a certain set of four imaginary quaternions is, of course, not without interest to him, but should be looked upon by him as a mere coincidence.

*See article on Quaternions by Professor Tait in *Encyclopedia Brittannica*.

CONDITION IN TERMS OF THE INVARIANTS OF THE QUARTIC THAT ITS FOUR DISTINCT ROOT-POINTS BE CONCYCLIC.

By DR. T. E. McKINNEY, Wesleyan University, Middletown, Conn.

The necessary and sufficient condition that the root-points of the quartic

$$(1) \qquad \sum_{i=0}^{i=4} a_i z^{4-i}, \ a_i \text{ complex},$$

be concyclic is that an anharmonic ratio of the roots of the quartic be real. The equation giving the six anharmonic ratios of these four roots is

$$(2) \qquad t^6-3t^5+\left(6-\frac{I^3}{\triangle}\right)t^4-\left(7-\frac{2I^3}{\triangle}\right)t^3+\left(6-\frac{I^3}{\triangle}\right)t^2-3t+1=0,$$

where $\qquad a_0^2 I = a_2^2 - 3a_1 a_2 + 12 a_0 a_4;$

$$a_0^3 I_1 = 27(a_1^2 a_4 + a_0 a_3^2) - 9a_1 a_2 a_3 - 72 a_0 a_2 a_4 + 2a_2^3,$$

$$27 a_0^6 \triangle = 64 I^3 - I_1^2.$$

The discriminant D of equation (2) is

$$D = \Pi (t_i - t_j)^2, \ i=1, 2, \ldots, 5; j=i+1, \ldots, 6,$$

$$= \frac{I^{12}}{\triangle^4}\left(4\frac{I^3}{\triangle}-27\right)^3.$$

Every root of equation (2) is a rational function with real coefficients of every other, so that the roots are either all real or all complex. When the roots are all real $I^3/\triangle$ is real. When the roots are not only real but also distinct, $D>0$. When $I^3/\triangle$ is real, and $D>0$, equation (2) has an even number of pairs of conjugate roots. Hence two and, therefore, all roots are real. This result may be expressed as follows:

THEOREM. *The necessary and sufficient condition that the four distinct root-points of the quartic*

$$\sum_{i=0}^{i=4} a_i z^{4-i}, \ a_i \textit{ complex},$$

be concyclic is that $\frac{4I^3}{\triangle} - 27 > 0$, *where* I *and* $\triangle$ *are the invariants of the quartic.*

SUMMATION OF CERTAIN INFINITE SERIES.

By W. J. GREENSTREET, M. A., F. R. A. S., Editor of The Mathematical Gazette, Stroud, England.

I note with some surprise that no solution of Problem 221 has yet (November, 1906) reached the MONTHLY. I therefore take the liberty of pointing out the method usually adopted in questions of this type, and embody in my remarks solutions of many similar questions taken from Todhunter and Hogg's Trigonometry, Hobson's Trigonometry (our best English work on the subject), and from various Cambridge Scholarship and other papers of recent years. The whole will, I hope, form a useful summary for those to whom the methods are unfamiliar.

1. To save space, we write the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ to ∞ in the form $\sum_{n=1}^{n=\infty}\frac{1}{n^2}$; $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ to ∞, in the form $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$.

The sign $\equiv$ is used for "identical with." We know that

$$\left.\begin{aligned}\frac{\sin\theta}{\theta}&=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\frac{\theta^6}{7!}+\ldots\equiv\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{\theta^{2n-2}}{(2n-1)!}\\ \text{and also }&=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\equiv\prod_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\pi^2}\right)\end{aligned}\right\}(A).$$

Taking logarithms of the two right hand expressions, and expanding in powers of θ, we may equate the coefficients of the respective powers of θ in the system (A).

Thus equating coefficients of θ^2 we have

$$(a)\qquad -\frac{\theta^2}{3!}=-\theta^2\left(\sum_{n=1}^{n=\infty}\frac{1}{n^2\pi^2}\right),\text{ whence }\sum_{n=1}^{n=\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Hence, if a, b, c, ... denote all the prime numbers except unity,

$$\begin{aligned}(b)\qquad &(1-a^{-2})^{-1}(1-b^{-2})^{-1}(1-c^{-2})^{-1}\ldots\\ &=(1+a^{-2}+a^{-4}+\ldots)(1+b^{-2}+b^{-4}+\ldots)(1+c^{-2}+c^{-4}+\ldots)\ldots\\ &=1+(a^{-2}+b^{-2}+c^{-2}+\ldots)+\ldots+(a^{-r}b^{-s}\ldots)^2+\ldots\\ &=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\ldots\textit{ ad infinitum}=\frac{\pi^2}{6}.\end{aligned}$$

We also have at once $\overset{\infty}{\underset{2}{\Pi}}\ (1-a^{-2})=\frac{6}{\pi^2}$.

Again equating coefficients of θ^4 in (A), we have

$$\frac{1}{5!}-\tfrac{1}{2}\left(\frac{1}{3!}\right)^2=-\frac{1}{2\pi^4}\sum_{n=1}^{n=\infty}\frac{1}{n^4},$$

and it follows that

(c) $$\sum_{n=1}^{n=\infty}\frac{1}{n^4}=\frac{\pi^4}{12}\left(\tfrac{1}{3}-\tfrac{1}{5}\right)=\frac{\pi^4}{90}.$$

Then, if a, b, c, ... are the prime numbers we have

(d) $$(1-a^{-4})^{-1}(1-b^{-4})^{-1}(1-c^{-4})^{-1}\dots$$

$$=(1+a^{-4}+a^{-8}+\dots)(1+b^{-4}+b^{-8}+\dots)(1+c^{-4}+c^{-8}+\dots)$$

$$=(1+a^{-4}+b^{-4}+\dots+(a^r\,b^s\,c^t\dots)^4+\dots$$

$$=\sum_{n=1}^{n=\infty}\frac{1}{n^4}=\frac{\pi^4}{90}\text{ by }(c).$$

Thus dividing (d) by (b) we have

(e) $\overset{\infty}{\underset{2}{\Pi}}\ (1+a^{-2})=\frac{15}{\pi^2}$, which is in No. 221, p. 190. And

$$\overset{\infty}{\underset{2}{\Pi}}\ (1+a^{-2})^{-1}\equiv\frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\cdot\ \dots=\frac{\pi^2}{15}.$$

We can connect up the series $\sum_{n=1}^{n=\infty}\frac{1}{n^2}$ with $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$, as follows:

Let $S=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots$

$S_1=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$

Then $S=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\dots+\frac{1}{2^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots\right)=S_1+\frac{1}{2^n}$.

(f) $\therefore S_1=\frac{2^n-1}{2^n}S$. So that if S be known, S_1 is known. For instance, if $n=2$, $S=\frac{1}{6}\pi^2$ by (a),

(g) $$\text{and } S_1 \equiv \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} = \frac{2^2-1}{2^2} S = \tfrac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

If $n=4$, $S=\frac{\pi^2}{90}$ by (c),

(h) $$\text{and } S_1 \equiv \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^4} = \frac{2^4-1}{2^4} S = \tfrac{15}{16} \cdot \frac{\pi^4}{90} \text{ by (c) } = \frac{\pi^4}{96}.$$

(i) $$\text{Consider } \sum_{n=1}^{n=\infty} (-1)^{n-1} \sum_{n=1}^{n=\infty} \frac{1}{n^2}.$$

$$\text{Here } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots - \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{24} \text{ by (g) and (a) } = \frac{\pi^2}{12}.$$

(j) Now consider the identity $\left[\sum_{n=1}^{n=\infty} \frac{1}{n^2}\right]^2 \equiv \sum_{n=1}^{n=\infty} \frac{1}{n^4} + 2 \sum \sum \frac{1}{p^2 q^2}$, where p and q are any of the numbers 1, 2, 3, ..., ∞.

Here $\left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{90} + 2 \sum \sum \frac{1}{q^2 q^2}$, by (a) and (d).

Then, $\sum \frac{1}{p^2 q^2} = -\frac{\pi^4}{180} + \frac{\pi^4}{72} = \frac{\pi^4}{120}$. Hence the sum of the series formed by multiplying together every two of the terms of the series $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots$, is $\frac{\pi^4}{120}$.

Now consider the identity,

$$\frac{\frac{1}{2}n.(n+1)}{(2n+1)^4} = \tfrac{1}{8}\left[\frac{1}{(2n+1)^2} - \frac{1}{(2n+1)^4}\right].$$

$$\text{Here } \sum_{n=1}^{n=\infty} \frac{\frac{1}{2}n(n+1)}{(2n+1)^4} = \tfrac{1}{8}\left[\sum_{n=1}^{n=\infty} \frac{1}{(2n+1)^2} - \sum_{n=1}^{n=\infty} \frac{1}{(2n+1)^4}\right]$$

$$= \tfrac{1}{8}\left[\frac{\pi^2}{8} - \frac{\pi^4}{96}\right] \text{ by (g) and (h) } = \frac{\pi^2}{64}\left[1 - \frac{\pi^2}{12}\right].$$

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

275. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Given the simultaneous equations $x^y-y^x=0$ and $y-x=a(a+1)^{1/a}$; find a solution which is real when $a>-1$.

Solution by A. H. HOLMES, Brunswick, Maine, and L. E. NEWCOMB, Los Gatos, California.

$x^y=y^x$... (1); $y-x=a(a+1)^{1/a}$... (2). Put $y=x^z$. Then in (1) $x^{x^z}=x^{zx}$. $\therefore x^z=zx$. From (2), $x(x^{z-1}-1)=a(a+1)^{1/a}$, or $x(z-1)=a(a+1)^{1/a}$. Take $(z-1)=a$. Then $x=(a+1)^{1/a}$, $z=a+1$, $y=zx=(a+1)^{(a+1)/a}$. Putting for y and x these values in (1),

$$(a+1)^{(1/a)\,(a+1)^{(a+1)/a}}=(a+1)^{[(a+1)/a](a+1)^{1/a}};$$

$$\frac{(a+1)^{a+1}}{a}\log(a+1)=\frac{a+1}{a}\times(a+1)^{1/a}\log(a+1);\text{ and}$$

$$\frac{a+1}{a}=\frac{a+1}{a}.$$

$\therefore x=(a+1)^{1/a}$ and $y=(a+1)^{(a+1)/a}$, which are real when $a>-1$.

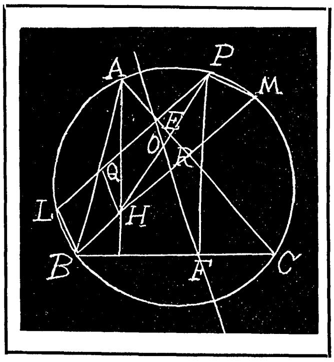
Also solved by G. B. M. Zerr and J. Scheffer.
No solution of 276 has yet been received.

GEOMETRY.

303. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove that the pedal line of any point on a triangle's circum-circle bisects the distance from this point to the triangle's ortho-center.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va., and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.



Let H be the ortho-center, P the given point, EF the pedal line intersecting PH in O. Produce the perpendicular BR to meet the circumference in M, and produce PE to meet the circumference in L, also take $EQ=PE$.

Quadrilateral $PEFC$ is inscriptible; $\angle PCF$ = supplement of angle PLB, both measured by $\frac{1}{2}$ arc PAB.

$\therefore \angle PEF$, the supplement of $\angle PCF=\angle PLB$. $\therefore EF$ is parallel to LB.

Now $RM=RH$, $PD=EQ$, hence the trape-

zoid $PQHM$ is isosceles. $\therefore PM=QH$, but $PM=LB$. $\therefore LB \doteq QH$ and is parallel to it since $\angle PLB=\angle PQH=\angle PEF$. $\therefore EF$ is parallel to QH.

Now E is the midpoint of PQ, hence O is the midpoint of PH.

Also solved by the Proposer.

304. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Find the tangent at the points $(a, 0)$ and $(0, a)$ to the locus $x^3+y^3=a^3$, and show that these points are points of inflexion.

I. Solution by A. H. HOLMES, Brunswick, Maine.

$x^3+y^3=a^3$. $\therefore \dfrac{dy}{dx}=-\dfrac{x^2}{(a^3-x^3)^{2/3}}$, which is 0 for $x=0$, and ∞ for $x=a$.

$\dfrac{d^2y}{dx^2}=-\dfrac{2x(a^3-x^3)^{1/3}-2x^4}{a^3-x^3}$, which is 0 for $x=0$, and ∞ for $x=a$. Take $x>a$,

and $\dfrac{d^2y}{dx^2}$ is seen to be minus. Take $x<a$ (a little) and $\dfrac{d^2y}{dx^2}$ is plus.

$\therefore (a, 0)$ and $(0, a)$ are points of inflexion.

II. Solution by BENJ. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

We have for the slope of the curve at any point, $\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$. $\therefore y-y_1=-\dfrac{x_1^2}{y_1^2}(x-x_1)$ is the equation of the tangent at any arbitrary point (x_1, y_1) of the curve. For $(0, a)$, the equation of the tangent is $y-a=0$. For $(a, 0)$, the equation of the tangent is $x-a=0$. From the equation of the tangent we have $y=y_1-\dfrac{x_1^2}{y_1^2}(x-x_1)$. In this equation, find y, for $x=x_1-h$ and $x=x_1+h$; also find the corresponding values of y from the equation of the curve, $y=\sqrt[3]{(a^3-x^3)}$. If the differences of these corresponding values of y change signs, the point is a point of inflection; if they do not, the point is an ordinary point of tangency. From the equation of the tangent, the values of y for the point $(0, a)$ are $y'=a$, $y''=a$, and from the curve $'y=\sqrt[3]{(a^3+h^3)}$, $''y=\sqrt[3]{(a^3-h^3)}$; $'y-y'=\sqrt[3]{(a^3+h^3)}-a>0$, $'y-y'=\sqrt[3]{(a^3-h^3)}-a<0$. Hence, $(0, a)$ is a point of inflection.

Similarly for the point $(a, 0)$, $y'=\infty$, $y''=-\infty$.
$'y=\sqrt[3]{(3a^2h-3ah^2+h^3)}$, $''y=\sqrt[3]{(-3a^2h-3ah^2-h^3)}$,
$'y-y'=\sqrt[3]{(3a^2h-3ah^2+h^3)}-\infty<0$, $''y-y''=\sqrt[3]{(-3a^2h-3ah^2-h^3)}+\infty>0$.

Hence, the point $(a, 0)$ is a point of inflection.

Also solved by G. B. M. Zerr, J. Scheffer, and the Proposer.

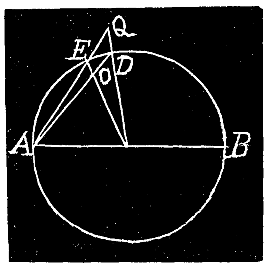
305. Proposed by JOHN J. QUINN, Ph. D., Scottdale, Pa.

1. Suppose two radii R and R_1 revolve uniformly in the ratio 2 : 3. Find the equation of the locus of the intersection of R with the chord drawn from the end of the diameter to the extremity of R_1. 2. If the chord be drawn to the extremity of R, what is the locus of the intersection with R_1? 3. Show how an angle can be trisected by means of this curve.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va., and A. H. HOLMES, Brunswick, Maine.

Let $CD=R$, $CE=R_1$. Then $\angle DCB=2\theta$, $\angle ECB=3\theta$, and O is the intersection of BE with R, while Q is the intersection of BD with R_1.

1. Let $CO=\rho$, $CB=a$. Then $CO:CB=\sin CBE:\sin COB$, or $\rho:a=\sin(\frac{1}{2}\pi-\frac{3}{2}\theta):\sin(\frac{1}{2}\pi-\frac{1}{2}\theta)$. $\therefore \rho=a\cos\frac{3}{2}\theta/\cos\frac{1}{2}\theta=a(1-4\sin^2\frac{1}{2}\theta)=a(2\cos\theta-1)$, for locus of O.



2. Let $CQ=\rho'$. $\therefore \rho':a=\sin(\frac{1}{2}\pi-\theta):\sin(\frac{1}{2}\pi-2\theta)$, and $\rho'=a\cos\theta/\cos2\theta$, for locus of Q.

3. Through any point O, on the curve, draw BO and produce it to meet the circumference in E. Then $\angle ECO=\frac{1}{3}\angle ECB$.

Through any point Q, on the curve, draw QB intersecting the circumference in D. Then $\angle QCD=\frac{1}{3}\angle QCB$.

CALCULUS.

231. Proposed by PROF. EVA S. MAGLOTT, Ada, Ohio.

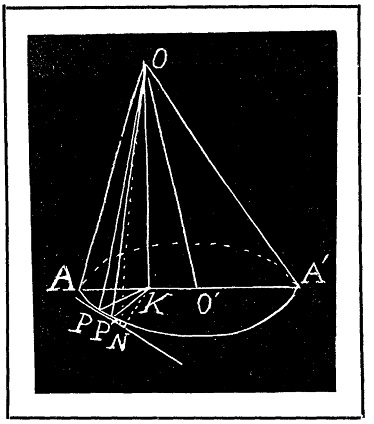
If a right circular cone stand on an ellipse, prove that the convex surface of the cone is $\frac{1}{2}\pi(OA+OA')(OA.OA')^{\frac{1}{2}}\sin\alpha$, where O is the vertex of the cone, A and A' the extremities of the major axis of the ellipse, and α is the semi-angle of the cone at the vertex, using the formula $ds=\frac{1}{2}\rho\sqrt{(\rho^2+p^2)}\,d\theta$, where p is the perpendicular from the vertex to the base of the cone, ρ the distance from the foot of the perpendicular to any point in the perimeter of the base, and θ the angle between the major axis and ρ.

Solution by BENJ. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

The problem as proposed assumes that the secant of the angle between an element of the convex surface and its projection on the base is $\sqrt{(p^2+\rho^2)}/\rho$. This assumption is incorrect, and therefore the convex surface cannot be found by means of the given formula. The correct formula is $ds=\frac{1}{2}\rho^2\,d\theta\sec\gamma$, where γ is the angle between an element of the convex surface and its projection on the base of the cone. The solution is as follows:

Let $O-APA'$ be the cone, axis OO', and OK the perpendicular from the vertex to the base. Let $\alpha=\angle A'OO'=\angle AOO'$; $\beta=\angle O'OK$; $\theta=\angle AKP$; $KO=p$; $KP=\rho$; OPP'= an element of the convex surface; and KPP' its pro-

jection on the base of the cone.



Area of $KPP'=\frac{1}{2}\rho^2 d\theta$; $\therefore$ area $OPP'=\frac{1}{2}\rho^2 d\theta \sec\gamma$. From the triangles $O'OP$ and $O'KP$, we have $O'O^2+OP^2-2O'O.OP\cos\alpha=O'K^2+KP^2+2O'K.KP\cos\theta$; whence, substituting and reducing, we have

$$\rho\sin\beta\cos\theta=p\cos\beta-\sqrt{(p^2+\rho^2)}\cos\alpha \ldots (1).$$

Tangent of angle, ψ, between ρ and the tangent PP' is $\frac{\rho d\theta}{d\rho}$. Let KN be the perpendicular from K on PN. Then angle $PKN=\psi-90°$.

$\therefore KN=\rho\cos(\psi-90°)=\rho\sin\psi$

$$=\rho.\frac{\rho d\theta}{\sqrt{(d\rho^2+\rho^2 d\theta^2)}}=\frac{\rho^2 d\theta}{\sqrt{(d\rho^2+\rho^2 d\theta^2)}}.$$

$$\sec\gamma=\frac{ON}{KN}=\sqrt{p^2+\left(\frac{\rho^2 d\theta}{\sqrt{(d\rho^2+\rho^2 d\theta^2)}}\right)^2}\div\frac{\rho^2 d\theta}{\sqrt{(d\rho^2+\rho^2 d\theta^2)}}$$

$$=\frac{1}{\rho^2}\sqrt{\rho^2\left(\frac{d\rho}{d\theta}\right)^2+\rho^2(p^2+\rho^2)}.$$

This must be taken with a negative sign, since $\sec\gamma$ diminishes as ρ increases.

$$\therefore dS=\tfrac{1}{2}\rho^2 d\theta\sec\gamma=-\tfrac{1}{2}\sqrt{\rho^2\left(\frac{d\rho}{d\theta}\right)^2+\rho^2(p^2+\rho^2)}\,d\theta \ldots (2).$$

From (1), $\dfrac{d\rho}{d\theta}=\dfrac{\rho\sqrt{(p^2+\rho^2)}\sin\beta\sin\theta}{\rho\cos\alpha+\sqrt{(p^2+\rho^2)}\sin\beta\cos\theta}$.

Substituting in (2) and reducing, we have $dS=$

$$-\tfrac{1}{2}p\sin\alpha\left[\frac{\rho d\rho}{\sqrt{[2p\sqrt{(p^2+\rho^2)}\cos\alpha\cos\beta-\rho^2(\cos^2\alpha-\sin^2\beta)-p^2(\cos^2\alpha+\cos^2\beta)]}}\right].$$

Let $p^2+\rho^2=x^2$. Substituting and reducing, we have

$$dS=-\tfrac{1}{2}p\sin\alpha\left[\frac{xdx}{\sqrt{[-\cos(\alpha-\beta)\cos(\alpha+\beta)x^2+2px\cos\alpha\cos\beta-p^2]}}\right].$$

The limits of ρ are AK and $A'K$. Hence, the limits of x are $\sqrt{(p^2+AK^2)}=AO$ and $\sqrt{(p^2+A'K^2)}=A'O$. Hence, the convex surface of the cone is

$$S=2\int_{AO}^{A'O}-\tfrac{1}{2}p\sin\alpha\left[\frac{xdx}{\sqrt{[-\cos(\alpha-\beta)\cos(\alpha+\beta)x^2+2px\cos\alpha\cos\beta-p^2]}}\right]$$

$$=-p\sin\alpha\left[\frac{\sqrt{[-\cos(\alpha-\beta)\cos(\alpha+\beta)x^2+2px\cos\alpha\cos\beta-p^2]}}{-\cos(\alpha-\beta)\cos(\alpha+\beta)}\right.$$

$$+\frac{p\cos\alpha\cos\beta}{\cos(\alpha-\beta)\cos(\alpha+\beta)}\left(\frac{1}{\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}\right.$$

$$\left.\left.\times\sin^{-1}\left(\frac{\cos(\alpha-\beta)\cos(\alpha+\beta)x-p\cos\alpha\cos\beta}{p\sin\alpha\sin\beta}\right)\right)\right]_{AO}^{A'O}$$

$$=-\frac{p^2\sin\alpha\cos\alpha\cos\beta}{\cos(\alpha-\beta)\cos(\alpha+\beta)\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}\left[\tfrac{1}{2}\pi-\tfrac{3}{2}\pi\right]$$

$$=\frac{p^2\sin\alpha\cos\alpha\cos\beta\,\pi}{\cos(\alpha-\beta)\cos(\alpha+\beta)\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}=\frac{A'O.AO\,\pi\sin\alpha\cos\alpha\cos\beta}{\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}.$$

Since $A'O\cos(\alpha+\beta)=AO\cos(\alpha-\beta)$, $\dfrac{A'O}{AO}=\dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$.

$$\frac{A'O+AO}{AO}=\frac{2\cos\alpha\cos\beta}{\cos(\alpha+\beta)}.\quad\text{Also}\ \frac{A'O+AO}{A'O}=\frac{2\cos\alpha\cos\beta}{\cos(\alpha-\beta)}.$$

$$\therefore\ \frac{(A'O+AO)^2}{A'O.AO}=\frac{4\cos^2\alpha\cos^2\beta}{\cos(\alpha-\beta)\cos(\alpha+\beta)},\ \text{or}\ \frac{\cos\alpha\cos\beta}{\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}$$

$$=\tfrac{1}{2}\frac{A'O+AO}{\sqrt{(A'O.AO)}}.$$

$$\therefore S=\tfrac{1}{2}\pi\sqrt{(A'O.AO)}\,(A'O+AO)\sin\alpha.$$

Dr. G. B. M. Zerr sent in two very simple solutions but not by projecting the convex surface on the plane of the elliptic base. Professor Scheffer sent in a solution similar to Dr. Zerr's. As the problem presents no difficulty when referred to rectangular axis with axis of the cone as one of the axes of coordinates, these solutions are omitted. The rectangular equation of the cone referred to planes $A'OA$, APA', and a plane through OK perpendicular to $A'OA$ is $y^2=\sec^2\alpha[(p-z)\sin(\alpha+\beta)+x\cos(\alpha+\beta)][(p-z)\sin(\alpha-\beta)-x\cos(\alpha-\beta)]$.

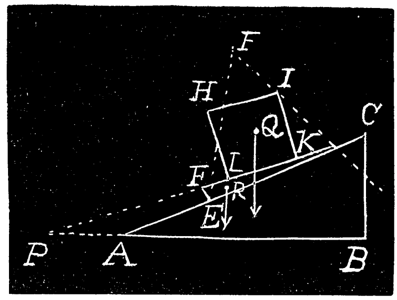
MECHANICS.

194. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A body has a plane face resting on a rough wedge. The wedge is on a rough inclined plane, thick end down and thin edge horizontal. Find the condition that the body will slide down the wedge with constant acceleration, the wedge not slipping the while. Discuss the case in which the angle of friction for wedge and plane is greater than the angle of inclination of the plane.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Constant acceleration is a force that produces a constant increase in velocity. Gravity is such a force.



Let $\angle CBA=\beta$, the angle of inclination of the plane; $\angle FDE=\delta$, the angle of the wedge. Let ν=coefficient of friction between wedge FDE and body $HIKL$.

If $\tan DPB=\tan(\beta-\delta)>\nu$, the constant acceleration is

$g[\sin(\beta-\delta)-\nu\cos(\beta-\delta)]$.

If $\angle FDE>\angle CAB$, then if $\tan(\delta-\beta)>\nu$, the constant acceleration is $g[\sin(\delta-\beta)-\nu\cos(\delta-\beta)]$.

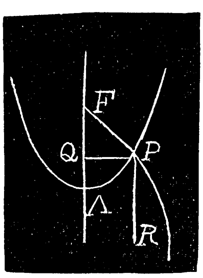
Hence, the condition for constant acceleration of the body on the wedge, the wedge not slipping, is that the coefficient of friction between body and wedge be less than the tangent of the difference of the angles of plane and wedge or wedge and plane. This covers the case for angle of friction for wedge and plane greater than angle of inclination of plane. The opposite case would form an interesting problem.

195. **Proposed by J. EDWARD SANDERS, Reinersville, Ohio.**

Particles slide from rest at the focus of a parabola, whose axis is vertical, down radius vectors, and are then allowed to move freely. Find the locus of the foci of their subsequent paths.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let PF be a radius vector$=r$, and $\angle PFA=\theta$. Let v be the velocity of the particle when leaving P. Draw PR parallel to AF. We find with reference to PR as the axis of abscissa, and P as origin of orthogonal co-ordinates, the equation



$$x=\frac{\cos\theta}{\sin\theta}y+\frac{1}{2v^2\sin^2\theta}y^2;$$

but $v^2=2gr\cos\theta$, and $r=\dfrac{a}{\cos^2\frac{1}{2}\theta}$. $\quad\therefore v^2=\dfrac{2ag\cos\theta}{\cos^2\frac{1}{2}\theta}$.

$\therefore$ Equation of curve

$$y^2+\frac{4a\sin\theta\cos^2\theta}{\cos^2\frac{1}{2}\theta}y=\frac{4a\cos\theta\sin^2\theta}{\cos^2\frac{1}{2}\theta}x.$$

From this equation it follows that the co-ordinates of the vertex of this

parabola with reference to its axis are $\frac{a\cos^3\theta}{\cos^2\frac{1}{2}\theta}$ and $\frac{2a\sin\theta\cos^2\theta}{\cos^2\frac{1}{2}\theta}$, and its parameter $\frac{a\sin^2\theta\cos\theta}{\cos^2\frac{1}{2}\theta}$.

Denoting by x and y the current co-ordinates of this curve with reference to the axis of the given parabola, and the focus as origin. we find from the above

$$x=\frac{a\cos\theta+a\sin^2\theta\cos\theta-a\cos^3\theta}{\cos^2\frac{1}{2}\theta}=\frac{a\sin2\theta\sin\theta}{\cos^2\frac{1}{2}\theta},$$

$$y=a\frac{\sin\theta-\sin2\theta\cos\theta}{\cos^2\frac{1}{2}\theta}=-\frac{a\sin\theta\cos2\theta}{\cos^2\frac{1}{2}\theta}.$$

By eliminating θ we get $(x^2+y^2-4a^2)^2(x^2+y^2)=y^2(x^2+y^2+4a^2)^2$. Introducing polar co-ordinates ρ and ϕ, we get the simpler equation

$$\rho=\pm2a\cot(45^\circ-\tfrac{1}{2}\phi) \text{ and } \rho=\pm2a\tan(45^\circ-\tfrac{1}{2}\phi).$$

Also solved by G. B. M. Zerr.

AVERAGE AND PROBABILITY.

160. **Proposed by J. F. LAWRENCE, A. M., Stillwater, Oklahoma.**

Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions. Find the mean value of that portion containing the center of gravity.

Solution by HENRY HEATON, Belfield, N. D.

The triangle may be considered equilateral (see Williamson's *Integral Calculus*, p. 355). Put $\triangle$ =area of equilateral triangle whose side$=a$. Let P be one of the points, $PD=y$ and $BD=x$. Let EPF be the line through the two points, and put $\angle BFP=\theta$ and $\angle BEP=\phi$. Then $\phi=\frac{2}{3}\pi-\theta$, $PF=y\operatorname{cosec}\theta$ and $DF=y\cot\theta$. The area of the elemental triangle $PFF'=\frac{1}{2}y^2\operatorname{cosec}^2\theta d\theta$. We will suppose the second point to be confined to this elemental triangle. Put $BF=z$. Then $z=x+y\cot\theta$, $BE=z\sin\theta\operatorname{cosec}\phi$, and area of triangle $EBF=\frac{\triangle z^2}{a^2}\sin\theta\operatorname{cosec}\phi$.

If $\theta<\frac{1}{6}\pi$ and F is confined to the line BC the area of the portion of the triangle containing the center of gravity is $\frac{\triangle}{a^2}(a^2-z^2\sin\theta\operatorname{cosec}\theta)$. The limits of z for this are $\frac{2}{3}y\sqrt{3}\sin\phi\operatorname{cosec}\theta$ and a. Those of y are 0 and $\frac{1}{2}a\sqrt{3}\sin\theta\operatorname{cosec}\phi$. If $\theta>\frac{1}{6}\pi$ and $<\frac{1}{3}\pi$, the line EF passes through O when $z=\frac{a}{3}(1+\sin\phi\operatorname{cosec}\theta)$.

If z is less than this, y is less than $\frac{a}{6}\sqrt{3}(1+\sin\theta\,\text{cosec}\phi)$.

If $z>\frac{a}{3}(1+\sin\phi\,\text{cosec}\theta)$ and $y<\frac{a}{6}\sqrt{3}(1+\sin\theta\,\text{cosec}\phi)$ the limits of z are $\frac{a}{3}(1+\sin\phi\,\text{cosec}\theta)$ and a.

$$\varDelta=\dot{\triangle}\Big[\int_0^{\frac{1}{6}\pi}\tfrac{1}{32}\sin\theta\,\text{cosec}^3\phi-\tfrac{1}{48}\sin^2\theta\,\text{cosec}^4\phi+\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\tfrac{1}{48}\sin^2\theta\,\text{cosec}^4\phi-\frac{1}{2^{\cdot}.3^7}$$
$$\times(\sin^2\theta\,\text{cosec}^4\phi+\sin^2\phi\,\text{cosec}^2\theta)+\frac{1}{2^5.3^6}(\sin\theta\,\text{cosec}^3\phi+\sin\phi\,\text{cosec}^3\theta+\frac{41}{2^4,3^7}$$
$$\times\text{cosec}\theta\,\text{cosec}\phi\Big]d\theta\div\tfrac{1}{32}\int_0^{\frac{1}{3}\pi}\sin\theta\,\text{cosec}^3\phi\,d\theta=\frac{\triangle}{3^6}(470+\tfrac{41}{3}\log2)=.6997\triangle.$$

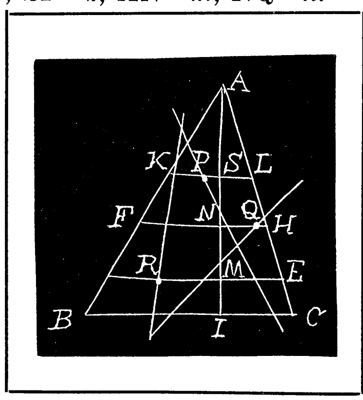
This is problem 76, p. 513, Williamson's *Integral Calculus*.

170. Proposed by LON C. WALKER, A. M., Santa Barbara, California.

Find the area of a triangle formed by drawing a line at random through each of three points taken at random within a given triangle.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Let ABC be the given triangle; P, Q, R the random points; $AI=h$, $BI=d$, $CI=e$, $d+e=a$, $AM=u$, $MR=v$, $AS=w$, $SP=z$, $AN=m$, $NQ=n$.



$y-v=r(x-u)$, the line through R...(1),
$y-z=s(x-w)$, the line through P...(2),
$y-n=t(x-m)$, the line through Q...(3),
where $r=\tan\theta$, $s=\tan\phi$, $t=\tan\psi$.

The intersection of (1) and (2) is given by

$$x_1=\frac{ru-ws+z-v}{r-s},\quad y_1=\frac{rsu-rsw+rz-sv}{r-s}.$$

The intersection of (1) and (3) is given by

$$x_2=\frac{ru-mt+n-v}{r-t},\quad y_2=\frac{rtu-mrt+rn-tv}{r-t}.$$

The intersection of (2) and (3) is given by

$$x_3=\frac{sw-mt+n-z}{s-t},\quad y_3=\frac{stw-mst+sn-tz}{s-t}.$$

Area of triangle$=\frac{1}{2}(x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_1y_3-x_3y_1)$
$=\frac{1}{2}[(ur-v)(s-t)+(ws-z)(t-r)+(mt-n)(r-s)]^2/(r-s)(r-t)(s-t)$
$=A/B$.

The limits of u are 0 and h; of w, 0 and u; of m, w and u; of v, $-du/h=v_1$ and $eu/h=v_2$; of z, $-dw/h=w_1$ and $ew/h=w_2$; of n, $-dm/h=m_1$ and $em/h=m_2$. The number of ways the three points can be taken on the surface of the triangle is $\frac{1}{6}(\frac{1}{2}ah)^3=\frac{1}{48}a^3h^3$.

Hence if $\triangle$ = the required average area we get

$$\triangle = \frac{48}{a^3 h^3}\int_0^h \int_0^u \int_w^u \int_{v_1}^{v_2} \int_{w_1}^{w_2} \int_{n_1}^{n_2} \frac{A}{B} du\, dw\, dm\, dv\, dz\, dn$$

$$= \frac{8}{ah^8 B}\int_0^h \int_0^u \int_w^u \{(s-t)^2[3ah^2r^2+3hr(d^2-e^2)+d^3+e^3]u^3wm$$

$$+(t-r)^2[3ah^2s^2+3hs(d^2-e^2)+d^3+e^3]uw^3m$$

$$+(r-s)^3[3ah^2t^2+3ht(d^2-e^2)+d^3+e^3]uwm^3\}du\, dw\, dm$$

$$+\frac{12}{a^2h^8 B}\int_0^h \int_0^u \int_w^u [(s-t)(t-r)(2ahr+d^2-e^2)(2ahs+d^2-e^2)u^2w^2m$$

$$+(s-t)(r-s)(2ahr+d^2-e^2)(2aht+d^2-e^2)u^2wm^2$$

$$+(t-r)(r-s)(2aht+d^2-e^2)(2ahs+d^2-e^2)uw^2m^2]du\, dw\, dm$$

$$= \frac{1}{24\,a}\left[\frac{9\,ah^2r^2(s-t)}{(r-s)\,(r-t)}+\frac{9\,hr(s-t)\,(d^2-e^2)}{(r-s)\,(r-t)}+\frac{3(d^3+e^3)\,(s-t)}{(r-s)\,(r-t)}\right.$$

$$+\frac{3\,ah^2s^2(r-t)}{(r-s)\,(s-t)}+\frac{3\,hs(r-t)\,(d^2-e^2)}{(r-s)\,(s-t)}+\frac{(d^3+e^3)\,(r-t)}{(r-s)\,(s-t)}$$

$$\left.+\frac{6\,ah^2t^2(r-s)}{(s-t)\,(r-t)}+\frac{6ht(r-s)\,(d^2-e^2)}{(s-t)\,(r-t)}+\frac{2(d^3+e^3)\,(r-s)}{(s-t)\,(r-t)}\right]$$

$$+\frac{1}{60a^2}\left[\frac{24\,a^2h^2rs}{s-r}+\frac{36\,a^2h^2rt}{r-t}+\frac{20\,a^2h^2st}{t-s}+\frac{12ah(d^2-e^2)\,(r+s)}{s-r}\right.$$

$$+\frac{18ah(d^2-e^2)\,(r+t)}{r-t}+\frac{10\,ah(d^2-e^2)\,(s+t)}{t-s}+\frac{6\,(d^2-e^2)}{s-r}+\frac{9(d^2-e^2)^2}{r-t}$$

$$\left.+\frac{5(d^2-e^2)^2}{t-s}\right]=\frac{1}{24\,a}C+\frac{1}{60\,a^2}D.$$

The limits of θ, ϕ, and ψ are for each 0 and $\frac{1}{2}\pi$ and doubled.

Now $\dfrac{r^2(s-t)}{(r-s)\,(r-t)}=\dfrac{r^2}{r-s}-\dfrac{r^2}{r-t}.$

$$\therefore \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{r^2(s-t)}{(r-s)\ (r-t)} d\theta\, d\phi\, d\psi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{s^2(r-t)}{(r-s)\ (s-t)} d\theta d\phi d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{t^2(r-s)}{(s-t)\ (r-t)} d\theta\, d\phi\, d\psi = 0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{r(s-t)}{(r-s)\ (r-t)} d\theta\, d\phi\, d\psi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{s(r-t)}{(r-s)\ (s-t)} d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{t(r-s)}{(s-t)\ (r-t)} d\theta\, d\phi\, d\psi = 0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{(s-t)}{(r-s)\ (r-t)} d\theta\, d\phi\, d\psi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{(r-t)}{(r-s)\ (s-t)} d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{(r-s)}{(s-t)\ (r-t)} d\theta\, d\phi\, d\psi = 0. \quad \therefore C = 0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{rs}{r-s} d\theta\, d\phi\, d\psi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{rt}{r-t} d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{st}{t-s} d\theta\, d\phi\, d\psi = \tfrac{1}{8}\pi^2.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{r+s}{r-s} d\theta\, d\phi\, d\psi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{r+t}{r-t} d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{s+t}{t-s} d\theta\, d\phi\, d\psi = 0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta\, d\phi\, d\psi}{s-r} = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta\, d\phi\, d\psi}{r-t} = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta\, d\phi\, d\psi}{t-s} = -\tfrac{1}{8}\pi^2.$$

$$\therefore D = \frac{10\pi^2 a^2 h^2 - \frac{5}{2}\pi^2 (d^2 - e^2)^2}{\frac{1}{8}\pi^3}.$$

$$\therefore \triangle = \frac{4\, a^2 h^2 - (d^2 - e^2)^2}{3\pi a^2} = \frac{[a^2(2c^2 + 2b^2 - a^2) - 2(c^2 - b^2)^2]}{3\pi a^2}.$$

If $a = b = c$, $\triangle = a^2/\pi$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

180. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find values of x, y, z, and u satisfying the equations

$$x+y+z+u=10\ldots[1],$$
$$x^2+y^2+z^2+u^2=30\ldots[2],$$
$$x^3+y^3+z^3+u^3=100\ldots[3],$$
$$x^4+y^4+z^4+u^4=354\ldots[4].$$

181. Proposed by A. H. HOLMES, Brunswick, Maine.

Sum the series, $1+2^m+3^m+4^m+\ldots+n^m$.

182. Proposed by O. L. CALLICOTT, Gettysburg, S. Dak.

Find the value of $\sqrt{2}\,\sqrt[3]{2}\,\sqrt[4]{2}\,\sqrt[5]{2}\ldots\sqrt[1000]{2}$.

GEOMETRY.

313. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that an algebraic curve of odd degree which is symmetrical with respect to a center has the center on the curve.

314. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Find the area of the triangle bounded by the lines $l\alpha+m\beta+n\gamma=0$; $l'\alpha+m'\beta+n'\gamma=0$; $l''\alpha+m''\beta+n''\gamma=0$, where α stands for $x\cos\alpha+y\sin\alpha-p$, etc. [See Salmon's *Conic Sections*, 6th ed.]

315. Proposed by ROBERT E. MORITZ, Ph. D., University of Washington.

Given the area of the segment of a circle of given radius to find the length of the chord.

CALCULUS.

237. *Prize Problem. Proposed by S. A. COREY, Hiteman, Iowa.

Find an expression for the remainder after n terms in the following development† of $f(a+x)$:

$$f(a+x)=f(a)+\frac{x}{m.2}\Big\{f'(a+x)+f'(a)+2\Big[f'\Big(a+\frac{x}{m}\Big)+f'\Big(a+\frac{2x}{m}\Big)$$
$$+f'\Big(a+\frac{3x}{m}\Big)+\ldots+f'\Big(a+\frac{m-1}{m}x\Big)\Big]\Big\}-\frac{B_1x^2}{m^2.2!}\Big[f''(a+x)-f''(a)\Big]$$
$$+\frac{B_2x^4}{m^4.4!}\Big[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\Big]-\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}\Big[f^{(2n)}(a+x)-f^{(2n)}(a)\Big]$$
$+\ldots$, B_1, B_2, ... being Bernoulli's numbers.

*In order to emphasize the importance of finding such an expression for the remainder after n terms as will hold good for all integral values of m and approach 0 as m approaches ∞, and at the same time enable computers to determine absolutely in what cases the development holds, the proposer offers a prize of $10 for the best solution. ED. F.

†See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

238. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equations

$$(a)\ (x+y^2)dx+(x^2+y)dy=0,$$
$$(b)\ (x+xy+y^2)dx-(x^2+xy-y)dy=0.$$

MECHANICS.

200. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

An elastic string whose weight is W is laid over the top of an inclined plane so as to remain at rest. Determine how much the string will be elongated, knowing, M=modulus of elasticity, L=normal length of string, and ϕ=inclination of the plane.

201. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

ABC is an inclined plane, perfectly rough, length $AC=l$. The time for a sphere to roll down when AB is base is to the time for a cylinder to roll down when BC is base as m is to n. Find AB and BC.

AVERAGE AND PROBABILITY.

187. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

188. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given (a) sphere, (b) cube.

MISCELLANEOUS.

169. Proposed by E. D. ROE, Ph. D., Syracuse University, Syracuse, N. Y.

Find the value for all finite values of k of

$$\operatorname*{Lim.}_{x \doteq \infty}\left[x^k \log\left(\frac{e^x+1}{e^x-1}\right)\right].$$

170. Proposed by J. W. NICHOLSON, A. M., LL. D., Baton Rouge, La.

If n and m are any two real numbers whatever, n being less than m, find a rational r such that $\sqrt{n}<r<\sqrt{m}$.

NOTES AND NEWS.

George William Greenwood has been appointed professor of mathematics at Roanoke College, Salem, Va. S.

The University of Washington, Seattle, announces the establishment of five teaching fellowships in mathematics, each yielding annual stipends from $400 to $500. Fellowships are open to graduate students only.

The twenty-first regular meeting of the Chicago Section of the American Mathematical Society will be held at The University of Chicago, Saturday, March 30, 1907. Abstracts of papers to be presented at this meeting should be in the hands of the Secretary not later than March 15. S.

"Introduction to Infinitesimal Analysis" is a new book from the press of John Wiley and Sons, by Oswald Veblen and N. J. Lennes. It is a valuable contribution to the literature available in English for the student of Advanced Calculus. It will be reviewed in a later issue of the MONTHLY. S.

It will be of interest to present readers of the MONTHLY to know that Professor Leonard E. Dickson, the retiring editor, contributed the first mathematical paper to these columns, Vol. I, No. 1, pp. 7-11, January, 1894. Since that time his interest has been continuous and his contributions numerous. S.

A preliminary meeting, to organize a Rochester, N. Y., Section of the Association of Teachers of Mathematics in the Middle States and Maryland, was held on February 23. This is in line with the rapidly increasing interest throughout the country in the question of bettering the teaching of mathematics, and it is most gratifying to see both colleges and secondary schools co-operating in this good cause. S.

At the last meeting of the Mathematics Section of the Central Association of Science and Mathematics Teachers, an important report on the Teaching of Geometry was presented by a committee of five who had been at work on the subject during the preceding year. This report is now in press, and may be had by enclosing a two cent stamp to the Secretary, Miss Mabel Sykes, 438 57th Street, Chicago, Ill. S.

BOOKS AND PERIODICALS.

The Teaching of Mathematics in the Elementary and the Secondary Schools. By J. W. A. Young, Ph. D., Assistant Professor of the Pedagogy of Mathematics in The University of Chicago. 8vo Cloth, xviii+350 pages. Price, $1.50.

This volume constitutes a very valuable addition to the Pedagogical Literature of Mathematics. The book is just from the press, and we have had time to read only the first three chapters and make a cursory examination of the remainder of the work. We have found the book so interesting and suggestive that we hasten to call the attention of our readers to it. The author tells of conditions as they are and states facts that are easily verified by investigation and experience. Too many works on general pedagogy describe conditions that are purely Utopian and that can never be realized so long as there are human beings to be educated. Thus, the author says, p. 11, "Some pupils no doubt regard the whole process of education (or any particular subject) as a set of tasks intended in some undefined way for the gratification of others, and consider that their own best interests lie in evading as far as possible the execution of these tasks." What teachers are there that have not observed this fact? Yet there are many so called teachers that refuse to acknowledge it. What teacher has not observed that most of the schools and colleges have just two classes of students? First, those who try to get as much out of their work as they can, and second, those who try to get out of as much of their work as they can. It is useless to try to write text books suitable for the latter class. Books composed of fly leaves would be the most satisfactory to them. The author also calls attention to the very patent fact that bad results in the teaching of mathematics are often due to the lack of special preparation on the part of the teacher. Thus, he says, p. 3, "It is even not unknown that classes in mathematics have been confided to teachers of other subjects, having neither special preparation for teaching mathematics nor experience in it, for no other reason than that they had a vacant period." It is a fact that teachers are too often selected without the slightest reference to their fitness or preparation for their work. This is true not only in mathematics but in other subjects as well.

The work contains the following fifteen chapters: The Study of the Pedagogy of Mathematics; The Purpose and Value of the Study of Mathematics in Primary and Secondary Schools; Methods and Modes; The Heuristic Method; The Individual Mode; The Perry Movement,—The Laboratory Method; Miscellaneous Points of Method and Mode; Preparation of Teachers,—Mathematical Clubs; The Material Equipment; The Curriculum in Mathematics; Definitions and Axioms; The Teaching of Arithmetic; The Teaching of Geometry; The Teaching of Algebra; Limits.

The true teacher of elementary mathematics will read this book with interest and profit. B. F. F.

Self Propelled Vehicles. A Practical Treatise on the Theory, Construction, Operation, Care, and Management of All Forms of Automobiles. By James E. Homans, A. M. 8vo Cloth, 598 pages. New York: Theo. Audel & Co.

This is the sixth edition of a work of great value to all interested in the automobile. It also contains much information of great value to the practical mechanic. B. F. F.

The Foundations of Higher Arithmetic. By B. F. Fisk, M. S., Instructor in Senior Arithmetic in the Austin High School. 8vo Cloth, vi+203 pages. New York: Silver, Burdett & Co.

This book is a step in the right direction of the teaching of the subject. The solutions are arranged in order, and are clear and concise. B. F. F.

The Elements of Geometry. By Walter N. Bush, Principal of the Polytechnic High School, San Francisco, and John B. Clarke, Department of Mathematics, Polytechnic High School, San Francisco. 8vo Cloth, x+355 pages. New York: Silver, Burdett & Co.

The treatment of Geometry in this book is along the line usually followed in American works. A large collection of originals are scattered throughout the book. B. F. F.

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SUMMATION OF CERTAIN INFINITE SERIES.

By W. J. GREENSTREET, M. A., F. R. A. S., Editor of The Mathematical Gazette, Stroud, England.

[Concluded from February Number.]

(l) To get a series such has $\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2}$, where m and n may have any integral values from 1 to ∞, we proceed as follows:

$$\left[\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}\right]^2=\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^4}+2\,\Sigma\Sigma\frac{1}{(2p+1)^2\,(2q+1)^2}.$$

$\therefore\ \Sigma\Sigma\frac{1}{(2p+1)^2\,(2q+1)^2}$ (where $2p+1$ and $2q+1$ are any odd numbers from 1 to ∞) $=\frac{1}{2}\left[\left(\frac{\pi^2}{8}\right)^2-\frac{\pi^4}{96}\right]$ by (g) and (h) $=\frac{\pi^4}{384}$.

(m) Consider the identity $\frac{1}{n^2(n+1)^2}\equiv\frac{1}{n^2}+\frac{1}{(n+1)^2}-2\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

$$\text{Here}\sum_{n=1}^{n=\infty}\frac{1}{n^2(n+1)^2}=\sum_{n=1}^{n=\infty}\frac{1}{n^2}+\left[\sum_{n=1}^{n=\infty}\frac{1}{n^2}-1\right]-2(1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}+\dots)$$

$$=\frac{\pi^2}{6}+\frac{\pi^2}{6}-1-2 \text{ by } (a) =\frac{\pi^2}{3}-3.$$

(n) Similarly, $\frac{1}{n^3\,(n+1)^3}\equiv\frac{1}{n^3}-\frac{1}{(n+1)^3}-\frac{3}{n^2\,(n+1)^2}$ gives

$$\sum_{n=1}^{n=\infty}\frac{1}{n^3(n+1)^3}=\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\dots\right)-\left(\frac{1}{2^3}+\frac{1}{3^3}+\dots\right)-3\sum_{n=1}^{n=\infty}\frac{1}{n^2\,(n+1)^2}$$

$$=1-3\left(\frac{\pi^2}{3}-3\right)=10-\pi^2, \text{ by } (m).$$

(*o*) In like manner, $\frac{1}{n^4 (n+1)^4} \equiv \frac{1}{n^4} + \frac{1}{(n+1)^4} - \frac{2}{n^2 (n+1)^2} - \frac{4}{n^3 (n+1)^3}$

gives $\sum_{n=1}^{n=\infty} \frac{1}{n^4 (n+1)^4} = \sum_{n=1}^{n=\infty} \frac{1}{n^4} + \left[\sum_{n=1}^{n=\infty} \frac{1}{n^4} - 1\right] - 2\sum_{n=1}^{n=\infty} \frac{1}{n^2 (n+1)^2}$

$-4\sum_{u=1}^{n=\infty} \frac{1}{n^3 (n+1)^3} = 2.\frac{\pi^4}{90} - 1 - 2\left(\frac{\pi^2}{3} - 3\right) - 4(10-\pi^2)$, by (*c*), (*m*), (*n*)

$$= \frac{\pi^4}{45} + \frac{10\pi^2}{3} - 35.$$

(*p*) The type $\frac{1}{1.2.3}, \frac{1}{2.3.4}, \frac{1}{3.4.5}, \ldots$ suggests the identity

$$\frac{1}{n(n+1)(n+2)} \equiv \tfrac{1}{2}.\frac{1}{n} - \frac{1}{n+1} + \tfrac{1}{2}.\frac{1}{n+2}.$$

Squaring, we have

$$\frac{1}{[n(n+1)(n+2)]^2} \equiv \tfrac{1}{4}.\frac{1}{n^2} + \frac{1}{(n+1)^2} + \tfrac{1}{4}.\frac{1}{(n+2)^2} - \tfrac{3}{4}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Hence $\sum_{n=1}^{n=\infty} \frac{1}{[n(n+1)(n+2)]^2} = \tfrac{1}{4}.\frac{\pi^2}{6} + \left(\frac{\pi^2}{6} - 1\right) + \tfrac{1}{4}\left(\frac{\pi^2}{6} - 1 - \frac{1}{2^2}\right) - \tfrac{3}{4}.\tfrac{3}{2}$ by (*a*)

$$= \tfrac{1}{4}\pi^2 - \tfrac{39}{16}.$$

(*q*) Since $\frac{\sin\theta}{\theta} = \prod_{n=1}^{n=\infty}\left(1 - \frac{\theta^2}{n^2 \pi^2}\right)$, taking logarithms and differentiating, we have

$$\frac{\cos\theta}{\sin\theta} - \theta = -\frac{2\theta}{\pi^2}\sum_{n=1}^{n=\infty} \frac{1/n^2}{1 - (\theta^2/n^2\pi^2)}.$$

$$\therefore \frac{\theta\sin\theta - \cos\theta}{\sin\theta} = \frac{2\theta^2}{\pi^2}\left[\frac{1}{1^2}\left(1 + \frac{\theta^2}{1^2\pi^2} + \frac{\theta^4}{1^4\pi^4} + \ldots\right)\right.$$

$$\left. + \frac{1}{2^2}\left(1 + \frac{\theta^2}{2^2\pi^2} + \frac{\theta^4}{2^4\pi^4} + \ldots\right) + \ldots\right]$$

$$= 2\left[\frac{\theta^2}{\pi^2}.\Sigma_2 + \frac{\theta^4}{\pi^4}\Sigma_4 + \frac{\theta^6}{\pi^6}\Sigma_6 + \ldots\right].$$

$$\therefore \frac{\theta\sin\theta - \cos\theta}{2} = \sin\theta\left[\frac{\theta^2}{\pi^2}\Sigma_2 + \frac{\theta^4}{\pi^4}\Sigma_4 + \ldots\right]$$

$$= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)\left[\frac{\theta^2}{\pi^2}\Sigma_2 + \frac{\theta^4}{\pi^4}\Sigma_4 + \ldots\right].$$

Now equating the coefficients of θ^{2n+1}, we have, on expanding $\theta\sin\theta-\cos\theta$,

$$\frac{n\pi^{2n}}{(2n+1)!}=\Sigma_2.\frac{\pi^{2n-2}}{(2n-1)!}-\Sigma_4.\frac{\pi^{2n-4}}{(2n-3)!}+\ldots-(-1)^n\Sigma_{2n},$$

where $\Sigma_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$ to ∞.

Hence for $n=1$, $\Sigma_2=\frac{\pi^2}{3!}$, which is (a);

$n=2$, $\Sigma_2.\frac{\pi^2}{3!}-\Sigma_4=\frac{2\pi^4}{5!}$ and $\Sigma_4=\frac{\pi^4}{90}$, which is (c);

$n=3$, $\Sigma_2.\frac{\pi^4}{5!}-\Sigma_4.\frac{\pi^2}{3!}+\Sigma_6.\frac{1}{1!}=\frac{3\pi^6}{7!}$.

$$\therefore \Sigma_6=\frac{-\pi^6}{3!\,5!}+\frac{\pi^6}{90.3!}+\frac{3\pi^6}{7!}=\frac{\pi^6}{945}.$$

(r) If 1, a, b, c, ..., be all the prime numbers, the numbers not divisible by the square of any prime (unity excepted) are a, b, c, ..., ab, ac, ..., abc, ... Then, the sum of the squares of the reciprocals of all numbers not divisible by the square of any prime is

$$1+a^{-2}+b^{-2}+c^{-2}+\ldots+a^{-2}b^{-2}+\ldots+a^{-2}b^{-2}c^{-2}+\ldots=\Pi(1+a^{-2})=15/\pi^2 \text{ by } (e).$$

(s) *Lemma.* To find the limit of $\frac{\sin\theta}{1-(\theta^2/\pi^2)}$ when $\theta=\pi$. Put $\theta=\pi+h$.

$$\frac{\sin\theta}{1-(\theta^2/\pi^2)}=\frac{\sin h}{(2\pi h+h^2)/\pi^2}=\frac{\sin h}{h}.\frac{1}{2/\pi+h/\pi^2}=\tfrac{1}{2}\pi \text{ for } h=0.$$

Now $\frac{\sin\theta}{\theta}=\prod_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\pi^2}\right)$. On dividing both sides by all factors of the type $1-\frac{\theta^2}{\lambda^2\pi^2}$ where λ is a prime, we have

$$\frac{\sin\theta}{\theta[1-(\theta^2/\pi^2)][1-(\theta^2/2^2\pi^2)]\ldots}=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\ldots$$

Now if $\theta=\pi$ this becomes

$$\tfrac{1}{2}\left(1-\frac{1}{2^2}\right)^{-1}\left(1-\frac{1}{3^2}\right)^{-1}\ldots=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\ldots$$

The left hand $=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots\right)\ldots$

$$=\tfrac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{12} \text{ by } (a).\right.$$

Taking logarithms of both sides

$$2\log\pi-\log 12=\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots+\tfrac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\ldots\right)+\tfrac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\ldots\right)+\ldots$$

$$=\Sigma_2+\tfrac{1}{2}\Sigma_4+\tfrac{1}{3}\Sigma_6+\ldots$$

where S_r is the sum of the reciprocals of the rth powers of all numbers that are not prime.

(t) Equate coefficients of θ^8 in (A)

$$-\frac{1}{4\pi^8}\Sigma_8=\frac{1}{9!}-\tfrac{1}{2}\left(\frac{1}{5!\,5!}+\frac{2}{3!\,7!}\right)+\tfrac{1}{3}\cdot\frac{3}{3!\,3!\,5!}-\tfrac{1}{4}\frac{1}{(3!)^4}=-\tfrac{1}{4}\cdot\frac{1}{9450}.$$

$$\therefore\ \Sigma_8=\frac{\pi^8}{9450}.$$

(u) $$\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{1}{n^4}\left\{\sum_{n=1}^{n=\infty}\frac{1}{n^4}\right\}\left(1-2.\frac{1}{2^4}\right)=\frac{\pi^4}{90}\cdot\frac{7}{8}=\frac{7\pi^4}{720}.$$

(v) $$\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{1}{n^8}\left(\sum_{n-1}^{n=\infty}\frac{1}{n^8}\right)\left(1-2.\frac{1}{2^8}\right)=\frac{7\pi^4}{720}\times\frac{127}{128}.$$

(w) $$\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^8}=\left(\sum_{n=1}^{n=\infty}\frac{1}{n^8}\right)\left(1-\frac{1}{2^8}\right)=\frac{\pi^8}{9450}\times\frac{255}{256}=\frac{17\pi^8}{256\times 630}.$$

(x) $$\sum_{n=1}^{n=\infty}\frac{4}{n^2(n+3)^2}=\tfrac{4}{9}\sum_{n=1}^{n=\infty}\left(\frac{1}{n^2}+\frac{1}{(n+3)^2}\right)$$

$$-\tfrac{8}{9}\sum_{n=1}^{n=\infty}\left(\frac{1}{n(n+1)}-\frac{2}{n(n+1)(n+2)}+\frac{2}{n(n+1)(n+2)(n+3)}\right)$$

$$=\tfrac{4}{9}\left(\frac{\pi^2}{6}+\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}\right)-\tfrac{8}{9}\left(1-\frac{2}{4}+\frac{2}{18}\right)=\frac{4\pi^2-31}{27}.$$

(y) $$\frac{2}{1^3+2^3}+\frac{3}{3^3+4^3+5^3}+\frac{4}{6^3+7^3+8^3+9^3}+\ldots$$

$$=\sum_{n=1}^{n=\infty}\frac{8}{n^2(n+1)^2(n+2)^2}$$

$$=\sum_{n=1}^{n=\infty}\left(\frac{2}{n^2}+\frac{8}{(n+1)^2}+\frac{2}{(n+2)^2}-\frac{12}{n(n+1)}+\frac{12}{n(n+1)(n+2)}\right)$$

$$=\frac{12\pi^2}{6}-8-2\left(\frac{1}{1^2}+\frac{1}{2^2}\right)-12+3=2\pi^2-19\tfrac{1}{2}.$$

(z) $$\frac{1}{1^2\cdot 2^2}-\frac{3}{4^2\cdot 5^2}+\frac{5}{7^2\cdot 8^2}+\ldots=\tfrac{1}{3}\left[1-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\ldots\right]$$

$$=\tfrac{1}{3}\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}-\ldots\right]$$

$$-\frac{1}{3^3}\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\right]=\left(\frac{1}{3}-\frac{1}{3^3}\right)\frac{\pi^2}{12}\text{ by }(i)=\frac{2\pi^2}{81}.$$

(α) From the identity

$$\frac{1}{(4n-3)^4(4n-1)^4}=\frac{1}{2^4}\left[\frac{1}{(4n-3)^4}+\frac{1}{(4n-1)^4}\right]-\frac{1}{2^3}\left[\frac{1}{(4n-3)^3}-\frac{1}{(4n-1)^3}\right]$$
$$+\frac{5}{2^5}\left[\frac{1}{(4n-3)^2}+\frac{1}{(4n-1)^2}\right]-\frac{5}{2^5}\left[\frac{1}{4n-3}-\frac{1}{4n-1}\right]$$

we have

$$\frac{1}{1^4\cdot 3^4}+\frac{1}{5^4\cdot 7^4}+\frac{1}{9^4\cdot 11^4}+\ldots=\sum_{n=1}^{n=\infty}\frac{1}{(n-3)^4(n-1)^4}$$

$$=\frac{\pi^4}{96\times 2^4}-\frac{1}{2^3}\cdot\frac{\pi^3}{32}+\frac{5}{2^5}\cdot\frac{\pi^2}{8}-\frac{5}{2^5}\cdot\frac{\pi}{4}=\frac{1}{1536}(\pi^4-6\pi^3+30\pi^2-60\pi).$$

To find $\frac{1}{1^4\cdot 3^4}+\frac{1}{3^4\cdot 5^4}+\frac{1}{5^4\cdot 7^4}+\ldots=\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^4(2n+1)^4}$ we use the identity

$$\frac{1}{(2n-1)^4(2n+1)^4}=\frac{1}{2^4}(u^4+v^4)-\frac{1}{2^3}(u^3-v^3)+\frac{5}{2^5}(u^2-v^2)$$
$$-\frac{5}{2^5}(u-v)\text{ where }u=\frac{1}{2n-1},\ v=\frac{1}{2n+1}.$$

Hence the sum required is found to be $\frac{1}{768}(\pi^4+30\pi^2-384)$.

(β) Taking logarithms in (A) and equating coefficients of θ^4, and also of θ^8 we have $\sum_{n=1}^{n=\infty}\frac{1}{n^4}$ and $\sum_{n=1}^{n=\infty}\frac{1}{n^8}$ as in (c) and (t).

$$\text{Now }2\sum_{m=1}^{m=\infty}\frac{1}{m^4}\sum_{n=1}^{n=\infty}\frac{1}{n^4}=\left(\sum_{n=1}^{n=\infty}\frac{1}{n^4}\right)^2-\sum_{n=1}^{n=\infty}\frac{1}{n^8}$$

$$=\frac{\pi^8}{8100}-\frac{\pi^8}{9450}\text{ by }(c)\text{ and }(t)=\frac{2\pi^8}{113400}.$$

$$\therefore \sum_{m=1}^{m=\infty}\frac{1}{m^4}\sum_{n=1}^{n=\infty}\frac{1}{n^4}=\frac{384\pi^8}{9!\,5!}$$

and this is the sum of the products of the fourth powers of the reciprocals of every pair of positive integers.

(γ) Consider the identity $\dfrac{1}{n(n+1)(n+2)}=\dfrac{1}{2.n}-\dfrac{1}{n+1}+\dfrac{1}{2(n+2)}$.

We have $$\frac{1}{n^2(n+1)^2(n+2)^2}=\frac{1}{4}\cdot\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{4(n+2)^2}$$

$$-\frac{1}{n}\cdot\frac{1}{(n+1)}+\frac{1}{2}\cdot\frac{1}{n}\cdot\frac{1}{n+2}-\frac{1}{(n+1)(n+2)}=\ldots-\frac{3}{4}\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

$$\therefore \sum_{n=1}^{n=\infty}\frac{1}{n^2(n+1)^2(n+2)^2}=\tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{n^2}+\tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{(n+1)^2}+\tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{(n+2)^2}$$

$$-\tfrac{3}{4}(1-\tfrac{1}{3}+\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{4}-\tfrac{1}{6}-\ldots)$$

$$=\frac{\pi^2}{24}+\frac{\pi^2}{6}-1+\frac{1}{4}\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)-\frac{3}{4}\cdot\frac{3}{2}$$

$$=\frac{\pi^2}{4}-\frac{39}{16}.$$

(δ) Since, when a, b, c, ... are all the primes save unity

$\prod_2^\infty (1-a^{-8})^{-1}=\dfrac{\pi^8}{9450}$, and $\prod_2^\infty (1-a^{-4})^{-1}=\dfrac{\pi^4}{90}$ by (c) and (t)

$$\prod_2^\infty (1+a^{-4})^{-1}=\frac{\pi^4}{105},\ i.\ e.,\ \frac{2^4}{2^4+1}\cdot\frac{3^4}{3^4+1}\cdot\frac{5^4}{5^4+1}\cdots=\frac{\pi^4}{105}.$$

(ε) Since $\displaystyle\sum_{n=1}^{n=\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, $\displaystyle\sum_{u=1}^{n=\infty}\frac{1}{(3n)^2}=\frac{\pi^2}{54}$.

Subtracting, we have $\dfrac{4\pi^2}{27}=1+\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\ldots$ where the denominators are the squares of all numbers not divisible by 3,

Euler's *Introductio in Analysis Infinitorum* (1748) Vol. I, Caput X. *De usu Factorum inventorum in definiendis summis Serierum infinitarum,* will be found a happy hunting-ground for those who want more of these series. The Latin is easily comprehended, the reader soon getting used to *Dignitates, Potestates, Exponentes,* and the like, while the methods are worth studying carefully by all interested in the historical development of mathematics.

A NOTE CONCERNING MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.*

By DR. GILBERT AMES BLISS.

In the discussions of maxima and minima of functions of a single variable, it is usually assumed that the function $f(x)$ under consideration has n continuous derivatives in the neighborhood of a point a at which

$$f'(a)=f''(a)=\ldots=f^{(n-1)}(a)=0 \text{ while } f^{(n)}(a)\neq 0.\dagger$$

The proofs usually given of the corresponding theorems for functions of several variables demand the existence and continuity of order $n+1$, that is of an order one higher than that of the first derivatives which do not vanish at the point under investigation.‡ In the following paragraphs a method is given for functions of several variables in which only the derivatives of the first n orders are used, as in the case of a single independent variable.

Suppose that all the derivatives of the function $f(x_1, x_2, \ldots, x_k)$ up to and including those of the nth order exist and are continuous in a vicinity

$$V(a): \qquad a_i-\delta \leqq x_i \leqq a_i+\delta \quad (i=1, 2, \ldots, k),$$

of the point $(a_1, a_2, \ldots, a_k)$. If the derivatives of order less than n all vanish at $(a_1, a_2, \ldots, a_k)$ then by Taylor's formula

$$(1) \quad f(a_1+h_1, a_2+h_2, \ldots, a_k+h_k)-f(a_1, a_2, \ldots, a_k)$$
$$=\sum \frac{h_1^{\alpha_1}}{\alpha_1!}\cdot\frac{h_2^{\alpha_2}}{\alpha_2!}\cdots\frac{h_k^{\alpha_k}}{\alpha_k!}\cdot\frac{\partial^n f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_k^{\alpha_k}},$$

where the summation is for all sets of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ whose sum is n, and the arguments of the derivatives of f are $a_i+\theta h_i$ $(i=1, 2, \ldots, k)$, $0<\theta<1$. If the right member is divided by r^n, where

$$r=\sqrt{[h_1^2+h_2^2+\ldots+h_k^2]},$$

it becomes a homogeneous quadratic form H_n of degree n in the k ratios $\eta_i=\dfrac{h_i}{r}$ which satisfy the equation

$$(2) \qquad \eta_1^2+\eta_2^2+\ldots+\eta_k^2=1.$$

Equation (1) may therefore be written

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*Read at the Chicago Meeting of the American Mathematical Society, December 30, 1906.

†See for example, Pierpont, *The Theory of Functions of Real Variables*, p. 318.

‡Pierpont, *loc. cit.*, p. 322, *ff.*

$$f(a_1+h_1, a_2+h_2, \ldots, a_k+h_k)-f(a_1, a_2, \ldots, a_k)=r^n H_n(\eta, a+\theta h),$$

where η, $a+\theta h$ stand for the rows of letters $(\eta_1, \eta_2, \ldots, \eta_k)$ and $(a_1+\theta h_1, a_2+\theta h_2, \ldots, a_k+\theta h_k)$, respectively.

When the values $a+\theta h$ in H_n are replaced by the values a and the η's are allowed to take any values satisfying equation (2), the quadratic form $H_n(\eta, a)$ belongs to one of the following classes:

a) It is indefinite, *i. e.*, there exist values of η for which it has opposite signs; or

b) It is semi-definite, *i. e.*, it does not take on opposite signs but there exist some values of η for which it vanishes; or

c) It is definite, *i. e.*, for all values of η it has the same sign.

If the form $H(\eta, a)$ is indefinite, the function f can not have an extreme at the point $(a_1, a_2, \ldots, a_k)$.

The proofs of this statement and also of the theorems below depend upon the fact that the function $H(\eta, a+h)$ is a continuous function for all values of η and for all values of $h_1, h_2, \ldots, h_k$ defining points in the vicinity $V(a)$. Suppose that η' is a set of values for which $H(\eta', a)$ is negative. On account of its continuity $H(\eta, a+h)$ will also be negative for all points $a+h$ in a certain neighborhood $V'(a)$. The equations $h'_i=r\eta'_i$ may now be taken as defining the values h', and $r>0$ can be restricted so that all the points $a+h'$ lie in $V'(a)$. Then it is evident that for the points $a+h'$ the expression (1) is negative. In a similar way, by taking values of η'' at which $H(\eta'', a)>0$, points $a+h''$ can be found in any neighborhood of a such that the difference in (1) is positive.

If the form $H_n(\eta, a)$ is definite, then the function f has a minimum at the point $(a_1, a_2, \ldots, a_k)$ when H_n is positive, and a maximum when it is negative.

If $H_n(\eta, a)$ is definite, a neighborhood $V'(a)$ can be found such that $H_n(\eta, a+h)$ is also different from zero for all sets of values η satisfying (2), and for all points $a+h$ in $V'(a)$. This follows because $H_n(\eta, a+h)$ is continuous, and positive over the closed set of points (η, h) of $2k$ dimensions which satisfy the equation (2) and $h=0$. Hence in $V'(a)$ the difference in (1) is positive or negative according as $H_n(\eta, a)$ is positive or negative.

The function $H_n(\eta, a)$ cannot vanish for all sets of values η unless the derivatives of f of order n are all zero at the point a. It follows that when n is odd, $H_n(\eta, a)$ is either identically zero or indefinite, for if η' represents values for which it is different from zero, $H(-\eta', a)=-H(\eta', a)$. It is possible, then, to state the preceding results in the following form:

A function $f(x_1, x_2, \ldots, x_k)$ is given which with its first n derivatives is continuous in a certain neighborhood $V(a)$ of the point $(a_1, a_2, \ldots, a_k)$. The derivatives of the first $n-1$ orders are zero at the point $(a_1, a_2, \ldots, a_k)$, while sum at least of order n are different from zero. Then if n is odd, or if n is even and the form

$$H_n(\eta, a) = \sum \frac{\eta_1^{\alpha_1}}{\alpha_1!} \cdot \frac{\eta_2^{\alpha_2}}{\alpha_2!} \cdots \frac{\eta_k^{\alpha_k}}{\alpha_k!} \frac{\partial^n f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$$

is indefinite at the point $(a_1, a_2, \ldots, a_k)$, *the function* f *has neither a maximum nor a minimum at the point* $(a_1, a_2, \ldots, a_k)$. *If* n *is even and* $H_n(\eta, a)$ *is positive definite the function* f *has a minimum, while if* $H_n(\eta, a)$ *is negative definite* f *has a maximum.*

The case, η even and $H_n(\eta, a)$ semi-definite, as usual requires a more elaborate investigation.

PRINCETON, *December, 1906.*

DIVIDING BY ZERO.

By DR. G. A. MILLER.

Since the seventh century of our era the Hindus have considered division by zero,* and in the twelfth century the noted Hindu mathematician and astronomer Bhaskara gave the rule that the quotient obtained by dividing a number by zero is not changed by adding even a large number to it or by substracting a large number from it. The same thought is expressed in modern times by "the quotient obtained by dividing a number which is not zero by zero is infinite." This rule is commonly understood to mean, explained by Krishna, a commentator of Bhaskara, that if we divide a given number which is not zero by a number which is small the quotient may be made to exceed any finite number if the divisor is made sufficiently small.

According to this interpretation $a/0$ need not mean the same thing as $\frac{a}{b-b}$. The matter may be made perfectly clear by adopting the notation used by Professor Pierpont in his recent work on the Theory of Functions. If a and b are two distinct numbers then there is an infinity of numbers which do not differ any more from a than b does. The totality of these numbers are said to form the domain of a, whose norm is $a \sim b = \rho$. This totality or aggregate of numbers is denoted by $D_\rho(a)$. It is sometimes desirable to exclude a from its domain. In this case the domain is said to be deleted and it is denoted by $D_\rho{}^*(a)$. In particular, $D_\rho{}^*(0)$ means all the numbers which differ from zero by not more than ρ, with the exception of zero itself. By making ρ sufficiently small we obtain the aggregate of numbers which are commonly considered when we think of the meaning of $a/0$,

*Cf. Algebra with arithmetic and mensuration from the Sanscrit of Brahmegupta and Bhascara translated by H. T. Colebrooke. London, 1817, p. 137.

so that this symbol is merely an abbreviation of the more rational symbol $a/D_\rho*(0)$.

While all are aware that division by zero, which was allowed by some of the most eminent older mathematicians, Euler for example, has caused a great deal of confusion and hence is commonly ruled out in modern mathematics, there are still some such practices in elementary mathematics which are apt to lead to confusion. It is the aim of this note to point to two instances of this kind and to suggest a method of avoiding the difficulties.

The first of these may be illustrated by the equations

$$\frac{1}{x}+\frac{1}{y}=1, \quad x+y=xy.$$

The second of these is obtained by clearing the first of fractions. No one will question the geometric interpretation of the second as it is a common form of the equation of a hyperbola passing through the origin. This hyperbola contains only one point whose co-ordinates do not satisfy the former of the given equations, viz., the origin. As there is no other point whose co-ordinates satisfy this equation we have the interesting result that the loci of $\frac{1}{x}+\frac{1}{y}=1$ and $x+y=xy$ differ only with respect to a single point.

If the loci of these two equations were drawn accurately the microscope of highest power would not make it possible to observe any difference since such an instrument could not exhibit the lacuna caused by the missing point. Hence some might at first be inclined to believe that for practical purposes it would make no difference whether we should call these equations equivalent or not. That this is far from the truth follows directly from the solution of such systems of simultaneous equations as

$$\begin{array}{ll} \frac{1}{x}+\frac{1}{y}=1, & x+y=xy, \\ x=y, & x=y. \end{array}$$

It is evident that the former of these systems is satisfied by only one pair of values for x and y while the latter is satisfied by two pairs, viz., (0, 0), $(\frac{1}{2}, \frac{1}{2})$. Moreover, the method suggested in the elementary algebras for the solution of such a system as

$$\frac{a}{x}+\frac{b}{y}=1, \quad \frac{c}{x}+\frac{d}{y}=1,$$

would not give us a complete solution of the system if the distinction indicated in the preceding paragraph were not observed.

The other point to which we wish to call attention in this connection is that the trigonometric equation $\tan 90^\circ=\infty$ is open to serious objection, for

$\tan A = \frac{\sin A}{\cos A}$, and hence $\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$; but, as we cannot divide by 0, we must say that $\tan 90^\circ$ has no value unless we assign to it some arbitrary value indepently of the fact that $\tan A = \frac{\sin A}{\cos A}$. Suppose that A is equal to some number in $D_\rho{}^*(\frac{1}{2}\pi)$, where ρ represents a small number. As ρ is decreased the smallest possible value of $\tan A$ is increased and we can always select ρ in such a way that $\tan A$ must be larger than any arbitrary number. This fact may be expressed by the equation

$$\tan D_\rho{}^*(\tfrac{1}{2}\pi) = \infty \text{ when } \rho \doteq 0.$$

It seems very unfortunate that we should tell the student of elementary algebra that it is not permitted to divide by 0 and then in trigonometry tell him that $\tan\frac{\pi}{2} = \frac{1}{0} = \infty$. It appears to be much better to say that $\tan A$ has a meaning for all values of A except when $A = k\frac{1}{2}\pi$. For these special values, it has no meaning but it becomes indefinitely large when A is very nearly equal to a number of the form $k\frac{1}{2}\pi$. It is evident that similar remarks apply to $\cot A$, $\sec A$, and $\csc A$. Trigonometry appears to be in special need of being freed from the hazy terminology which appealed to the Hindu mind of a thonsand years ago. The main conclusions which have been reached in the above considerations are:

While we may divide by every number in $D_\rho{}^*(0)$ we are not allowed to divide by every number in $D_\rho(0)$. The two domains differ only with respect to one number and if they could be represented geometrically the microscope of highest power in existence would not reveal the lacuna in $D_\rho{}^*(0)$ caused by the omission of 0. If we multiply by a factor to clear of fractions and then make this factor equal to zero, it is equivalent to dividing by zero and hence this operation cannot be allowed. It is better to say that $a/0$ has no meaning in elementary mathematics than to assign it an arbitrary value since the student cannot appreciate the need of such arbitrary values until he is more mature. There are many to whom the ordinary treatment of the trigonometric functions of $k\frac{1}{2}\pi$ appear objectionable and it is probable that the introduction of the concepts of *domain* and *deleted domain* would tend towards greater clearness in this part of the trigonometry.

ON A CERTAIN QUARTIC CURVE WHICH MAY DEGENERATE INTO AN ELLIPSE.*

By R. D. CARMICHAEL.

The object of this paper is the discussion of a class of quartic curves whose equation may be put in the form

$$(1) \qquad \nu\sqrt{[x^2+y^2]} \pm \mu\sqrt{[x^2+(a-y)^2]} = \lambda.$$

When the equation is cleared of radicals the double sign $\pm$ can be made to disappear entirely. In the following discussion it will be shown that this curve consists of two ovals. As a limiting case, when $\mu=\nu$ one of the ovals vanishes and the other becomes an ellipse.

§1. Construction of the Locus by Continuous Motion.

Take two points A and B at a distance a apart, and let A be the origin of rectangular axes, and let the y-axis pass through B. Take the point P such that

$$(2) \qquad \mu.PB+\nu.PA=\lambda.$$

Then P is a point on the locus; for evidently it satisfies the equation when the double sign is taken *plus*. Hence the branch on which P is located may be defined as the locus of a point which moves so that the sum of μ times its distance from one point and ν times its distance from another is constant. We may therefore devise the following method for constructing this branch when μ and ν are commensurable.

Divide equation (1) by such common factor of μ and ν as will make the respective quotients m and n relatively prime integers, and let the quotient of λ divided by this factor be l. Then the equation becomes

$$(3) \qquad n\sqrt{[x^2+y^2]} \pm m\sqrt{[x^2+(a-y)^2]} = l.$$

For the point P we now have

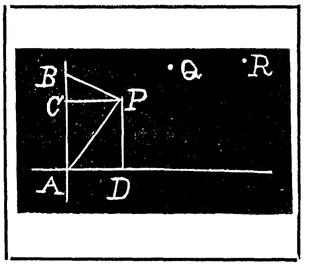
$$(4) \qquad m.PB+n.PA=l.$$

One or both of the numbers m, n are odd. Consider m odd and n even.

Attach a string at B and pass it around a pencil point at P, wrapping until m plies of the cord pass from B to P. Then pass the free end around A, wrapping until n plies pass from A to P. Then attach the cord to the

*Presented to the Chicago Section of the American Mathematical Society, March 30, 1907.

pencil at P so that every part of it is drawn tight. Now if the pencil so moves that the chord is kept tightly stretched while it slides around A, B, and P, P will describe one branch of the curve. (The modifications necessary for n odd or m even are evident). We shall next give a method for constructing the other branch by continuous motion. For this case we shall employ



(5) $$n\sqrt{[x^2+y^2]}-m\sqrt{[x^2+(a-y)^2]}=l.$$

If Q in the figure is so taken that

(6) $$n.AQ-m.BQ=l,$$

Q is evidently a point on this second branch.

Now, pass a cord around the point at B and the pencil at Q so that m plies pass from B to Q. If m is odd one end of the string is to be attached at B; while if m is even it is to be fastened to the pencil point at Q. The free end is to pass out in some direction as QR. In like manner join Q and A with another chord having its free end in the direction QR. Of this cord n plies are to pass from A to Q. Let these two chords be tightly stretched and tied together at some convenient point. If Q moves so as to keep these conditions always fulfilled, it is evident that the pencil will mark on the paper the second branch of this curve.

From the manner of construction it is evident that each of these branches is closed. Such a branch we shall as usual call an oval. In the next section we shall show that these two are the only branches of the locus.

§2. General Nature of the Locus.

As the two branches of the locus have the respective equations

$$\nu\sqrt{[x^2+y^2]}+\mu\sqrt{[x^2+(a-y)^2]}=\lambda,$$
$$\nu\sqrt{[x^2+y^2]}-\mu\sqrt{[x^2+(a-y)^2]}=\lambda,$$

it is evident they have no point in common except when

$$\mu\sqrt{[x^2+(a-y)^2]}=0.$$

Now μ is not zero; for then the locus would reduce to a circle. Hence

$$x^2+(a-y)^2=0.$$

Since each term of the first member is positive the equation can be satisfied only for $x^2=0$ and $(a-y)^2=0$. Hence, $x=0$, $y=a$. That is, the two

branches of the curve can have no point in common except $(0, a)$. Moreover, this point is on the locus only when $\nu a=\lambda$, as may be seen by substituting in equation (1). When this condition is satisfied the two branches are tangent at the point $(0, a)$; for if they should cut each other here they would necessarily have another point in common. Moreover one branch is always entirely within the other.

Now if there is another branch to the locus besides the two already discussed some line passing through the inner of these two ovals cuts this third branch. The line then intersects the *quartic* curve in *five* points; but this is impossible. Hence the entire locus consists of the two ovals above constructed.

PRESBYTERIAN COLLEGE, ANNISTON, ALABAMA.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

276. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

If $x_1, x_2, x_3, \ldots, x_n$ be unequal, and $f(x)$ be a rational integral function of degree $\not> n-2$, then shall

$$\sum_{r=1}^{r=n-1} \frac{f(x_r)}{(x_r-x_1)(x_r-x_2)\ldots(x_r-x_n)}=0.$$

No solution of this problem has been received. It is not clear what is intended in the problem. By taking $f(x)=x^2-x+1$ and $x_1=1$, $x_2=2$, $x_3=3$, $x_4=4$, the problem is not true according to our interpretation of it, since infinite addends would be introduced.

In Burnside and Panton's *Theory of Equations*, p. 319, 3rd edition, example 4, slightly changed, reads: If the degree of $\phi(x)$ does not exceed $n-2$, prove that $\sum_{r=1}^{r=n}\frac{\phi(x_r)}{f'(x_r)}=0$, ..., $x_r(r=1, 2, \ldots, n)$ being roots of $f(x)=0$.

Additional information on this problem is desired. ED. F.

277. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

If α, β, γ, δ are the roots of the quartic $ax^4+bx^3+cx^2+dx+e=0$, calculate the value of the product of the twelve expressions of the form $(4\alpha-$ $-2\beta-\gamma-\delta)$ in terms of H, I, J, the well known functions of the differences of the roots.

Solution by the PROPOSER.

To avoid fractions write the equation in the form $ax^4+4b'x^3+6c'x^2+4d'x+e=0$. See Burnside and Panton's *Theory of Equations*, 4th edition, Vol. I, pp. 121-131.

$H=ac'-b'^2$ $I=ae-4b'd'+3c'^2$, $J=ac'e+2b'c'd'-ad'^2-eb'^2-c'^3$.

$$\left.\begin{array}{l} M_1^2=a^2\theta_1+b'^2-ac'=a^2\theta_1-H \\ M_2^2=a^2\theta_2+b'^2-ac'=a^2\theta_2-H \\ M_3^2=a^2\theta_3+b'^2-ac'=a^2\theta_3-H \end{array}\right\}\ldots(1),$$

where θ_1, θ_2, θ_3 are the roots of the reducing cubic $4a^3\theta^3-Ia\theta+J=0$.

$$\left.\begin{array}{l} a\alpha+b'=-M_1+M_2+M_3 \\ a\beta+b'=M_1-M_2+M_3 \\ a\gamma+b'=M_1+M_2-M_3 \\ a\delta+b'=-M_1-M_2-M_3 \end{array}\right\}\ldots(2).$$

Now $\theta_1+\theta_2+\theta_3=0$, $\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3=-I/4a^2$, $\theta_1\theta_2\theta_3=-J/4a^3$.

The twelve expressions of the form $(4\alpha-2\beta-\gamma-\delta)$ can be easily expressed in terms of M_1, M_2, M_3 from (2), thus

$$\begin{array}{l}
\tfrac{1}{2}a(4\alpha-2\beta-\gamma-\delta)=-[3(M_1-M_2)-2M_3] \\
\tfrac{1}{2}a(4\alpha-\beta-2\gamma-\delta)=-[3(M_1-M_3)-2M_2] \\
\tfrac{1}{2}a(4\alpha-\beta-\gamma-2\delta)=[3(M_2+M_3)-2M_1] \\
\tfrac{1}{2}a(4\beta-2\alpha-\gamma-\delta)=[3(M_1-M_2)+2M_3] \\
\tfrac{1}{2}a(4\beta-\alpha-2\gamma-\delta)=-[3(M_2-M_3)-2M_1] \\
\tfrac{1}{2}a(4\beta-\alpha-\gamma-2\delta)=3(M_1+M_3)-2M_2 \\
\tfrac{1}{2}a(4\gamma-2\alpha-\beta-\delta)=3(M_1-M_3)+2M_2 \\
\tfrac{1}{2}a(4\gamma-\alpha-2\beta-\delta)=3(M_2-M_3)+2M_1 \\
\tfrac{1}{2}a(4\gamma-\alpha-\beta-2\delta)=3(M_1+M_2)-2M_3 \\
\tfrac{1}{2}a(4\delta-2\alpha-\beta-\gamma)=-[3(M_2+M_3)+2M_1] \\
\tfrac{1}{2}a(4\delta-\alpha-2\beta-\gamma)=-[3(M_1+M_3)+2M_2] \\
\tfrac{1}{2}a(4\delta-\alpha-\beta-2\gamma)=-[3(M_1+M_2)+2M_3]
\end{array}$$

These twelve can be reduced to three as follows: $\frac{1}{16}a^4(4\alpha-2\beta-\gamma-\delta)(4\beta-2\alpha-\gamma-\delta)(4\gamma-\alpha-\beta-2\delta)(4\delta-\alpha-\beta-2\gamma)$

$$\begin{aligned}
&=[9(M_1-M_2)^2-4M_3^2][9(M_1+M_2)^2-4M_3^2] \\
&=[9M_1^2+9M_2^2-4M_3^2-18M_1M_2][9M_1^2+9M_2^2-4M_3^2+18M_1M_2] \\
&=[9a^2\theta_1-9H+9a^2\theta_2-9H-4a^2\theta_3+4H-18M_1M_2]\ [9a^2\theta_1-9H+9a^2\theta_2 \\
&\qquad -9H-4a^2\theta_3+4H+18M_1M_2] \\
&=[9a^2(\theta_1+\theta_2+\theta_3)-13a^2\theta_3)-14H-18M_1M_2][9a^2(\theta_1+\theta_2+\theta_3) \\
&\qquad -13a^2\theta_3-14H+18M_1M_2] \\
&=(13a^2\theta_3+14H+18M_1M_2)(13a^2\theta_3+14H-18M_1M_2) \\
&=(13a^2\theta_3+14H)^2-324M_1^2M_2^2 \\
&=(13a^2\theta_3+14H)^2-324(a^2\theta_1-H)(a^2\theta_2-H)
\end{aligned}$$

$$=169a^4\theta_3^2+364a^2\theta_3H+196H^2-324a^4\theta_1\theta_2+324a^2\theta_1H+324a^2\theta_2H-324H^2$$
$$=169a^4\theta_3^2+40a^2\theta_3+324a^2H(\theta_1+\theta_2+\theta_3)-324a^4\theta_1\theta_2-128H^2$$
$$=169a^4\theta_3^2+40a^2\theta_3+81aJ/\theta_3-128H^2$$
$$=\frac{1}{\theta_3}(169a^4\theta_3^3+40a^2\theta_3^2H+81aJ-128H^2\theta_3)=\frac{1}{\theta_3}(A+81aJ).$$

Similarly,

$$\tfrac{1}{16}a^4(4\alpha-\beta-2\gamma-\delta)(4\beta-\alpha-\gamma-2\delta)(4\gamma-2\alpha-\beta-\delta)(4\delta-\alpha-2\beta-\gamma)$$
$$=\frac{1}{\theta_2}(169a^4\theta_2^3+40a^2\theta_2^2H+81aJ-128H^2\theta_2)=\frac{1}{\theta_2}(B+81aJ).$$
$$\tfrac{1}{16}a^4(4\alpha-\beta-\gamma-2\delta)(4\beta-\alpha-2\gamma-\delta)(4\gamma-\alpha-2\beta-\delta)(4\delta-2\alpha-\beta-\gamma)$$
$$=\frac{1}{\theta_1}(169a^4\theta_1^3+40a^2\theta_1^2H+81aJ-128H^2\theta_1)=\frac{1}{\theta_1}(C+81aJ).$$

Let Π= the product of the twelve factors.

$$\therefore\ \frac{a^{12}\Pi}{4096}=\frac{1}{\theta_1\theta_2\theta_3}[531441a^3J^3+6561a^2J^2(A+B+C)+81aJ(AB+AC+BC)+ABC]$$
$$=-\frac{4a^3}{J}[531441a^3J^3+6561a^2J^2(A+B+C)+81aJ(AB+AC+BC)+ABC].$$

$$A+B+C=169a^4(\theta_1^3+\theta_2^3+\theta_3^3)+40a^2H(\theta_1^2+\theta_2^2+\theta_3^2)-128H^2(\theta_1+\theta_2+\theta_3).$$

$$AB+AC+BC=28561a^8(\theta_1^3\theta_2^3+\theta_1^3\theta_3^3+\theta_2^3\theta_3^3)+1600a^4H^2(\theta_1^2\theta_2^2+\theta_1^2\theta_3^2+\theta_2^2\theta_3^2)+16384H^4(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)+6760a^6H(\theta_1^2\theta_2^3+\theta_1^3\theta_2^2+\theta_1^2\theta_3^3+\theta_1^3\theta_3^2+\theta_2^2\theta_3^3+\theta_2^3\theta_3^2)-21632a^4H^2(\theta_1^3\theta_2+\theta_1\theta_2^3+\theta_1^3\theta_3+\theta_1\theta_3^3+\theta_2^3\theta_3+\theta_2\theta_3^3)-5120a^2H^3(\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_1^2\theta_2+\theta_1^2\theta_3+\theta_1\theta_3^2+\theta_2^2\theta_3+\theta_2\theta_3^2).$$

$$ABC=4826809a^{12}\theta_1^3\theta_2^3\theta_3^3+64000a^6H^3\theta_1^2\theta_2^2\theta_3^2-2097152H^6\theta_1\theta_2\theta_3+270400a^8H^2\theta_1^2\theta_2^2\theta_3^2(\theta_1+\theta_2+\theta_3)+2768896a^4H^4\theta_1\theta_2\theta_3(\theta_1^2+\theta_2^2+\theta_3^2)+1142440a^{10}H\theta_1^2\theta_2^2\theta_3^2(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)-3655808a^8H^2\theta_1\theta_2\theta_3(\theta_1^2\theta_2^2+\theta_1^2\theta_3^2+\theta_2^2\theta_3^2)-204800a^4H^4\theta_1\theta_2\theta_3(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)+655360a^4H^4\theta_1\theta_2\theta_3(\theta_1+\theta_2+\theta_3)-865280a^6H^3\theta_1\theta_2\theta_3(\theta_1\theta_2^2+\theta_1^2\theta_2+\theta_1\theta_3^2+\theta_1^2\theta_3+\theta_2\theta_3^2+\theta_2^2\theta_3).$$

$$\theta_1^2+\theta_2^2+\theta_3^2=-2(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)=I/2a^2.$$

$$\theta_1^3+\theta_2^3+\theta_3^2=-3(\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_1^2\theta_3+\theta_1\theta_3^2+\theta_2^2\theta_3+\theta_2\theta_3^2)-6\theta_1\theta_2\theta_3=3\theta_1\theta_2\theta_3=-3J/4a^3.$$

$$\theta_1^2\theta_2^2+\theta_1^2\theta_3^2+\theta_2^2\theta_3^2=I^2/16a^4.$$

$$\theta_1^3\theta_2^3+\theta_1^3\theta_3^3+\theta_2^3\theta_3^3=-I^3/64a^6+3\theta_1\theta_2\theta_3=(12J^2-I^3)/64a^6.$$

$$\theta_1^2\theta_2+\theta_1\theta_2^2+\theta_1^2\theta_3+\theta_1\theta_3^2+\theta_2^2\theta_3+\theta_2\theta_3^2=-3\theta_1\theta_2\theta_3=3J/4a^3.$$

$$\theta_1^3\theta_2+\theta_1\theta_2^3+\theta_1^3\theta_3+\theta_1\theta_3^3+\theta_2^3\theta_3+\theta_2\theta_3^3=(I/2a^2)(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)=-I^2/8a^4.$$

$$\theta_1^3\theta_2^2+\theta_1^2\theta_2^3+\theta_1^3\theta_3^2+\theta_1^2\theta_3^3+\theta_2^3\theta_3^2+\theta_2^2\theta_3^3=-(\theta_1^2\theta_2^2\theta_3+\theta_1^2\theta_2\theta_3^2+\theta_1\theta_2^2\theta_3^2)$$

$$=-\theta_1\theta_2\theta_3(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)=-IJ/16a^5.$$

$$\therefore A+B+C=\tfrac{1}{4}(80HI-507aJ).$$

$$AB+AC+BC=\frac{1}{64a^2}(342732a^4J^2-28561a^4I^3+179456a^2H^2I^2-262144H^4I$$
$$-27040a^3HIJ-245760aH^3J).$$

$$ABC=\frac{1}{64a^3}(12943360a^3H^3J^2-4826809a^6J^3+33554432H^6J-22970368a^2H^4IJ$$
$$-1142440a^5HIJ^2+3655808a^4H^2I^2J).$$

$$\therefore \frac{a^{12}\Pi}{4096}=-[(2125764a^6J^2+524880a^5HIJ^2-3326427a^6J^2+\tfrac{1}{16}(27761292a^6J^2$$
$$-2313441a^6J^3+14535936a^4H^2I^2-21233664a^2H^4I-2190240a^5HIJ$$
$$-19906560a^3H^3J)+\tfrac{1}{16}(12943360a^3H^3J-4826809a^6J^2+33554432H^6$$
$$-22970368a^2H^4I-1142440a^5HIJ+3655808a^4H^2I^2).$$

$$\therefore a^{12}\Pi=256(2313441a^6I^3-3723875a^6J^2-5065400a^5HIJ+6963200a^3H^3J$$
$$+44204032a^2H^4I-18191744a^4H^2I^2-33554432H^6).$$

GEOMETRY.

306. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Find the length of the perpendicular let fall from the point in space 5, 6, 7) upon the line $x=2z-3$, and $y=-3z+1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $\left.\begin{matrix} x=sz+p \\ y=tz+q \end{matrix}\right\}$ be the line in space, and m, n, l the point in space. $\iota, n, l=A$, the vertex of a triangle; $p, q, 0=B$, the point where line pierces ıe xy plane; $\frac{pt-qs}{t}$, 0, $-\frac{q}{t}=C$, the point where line pierces the xz plane.

$c=AB=\sqrt{[(m-p)^2+(n-q)^2+l^2]}$.

$b=AC=\sqrt{\left(\frac{mt-pt+qs}{t}\right)^2+n^2+\left(\frac{lt+q}{t}\right)^2}$.

$a=BC=(q/t)\sqrt{(1+s^2+t^2)}$.

he perpendicular from the vertex A on the side BC is

$$p=\sqrt{c^2-\left(\frac{a^2+c^2-b^2}{2a}\right)^2}.$$

$$\therefore p=\sqrt{(m-p)^2+(n-q)^2+l^2-\left(\frac{(m-p)s+(n-q)t+l}{\sqrt{(1+s^2+t^2)}}\right)^2}$$

$$=\sqrt{\frac{(mt-pt-ns+qs)^2+(sl-m+p)^2+(tl-n+q)^2}{1+s^2+t^2}}.$$

In the problem, $m=5$, $n=6$, $l=7$, $s=2$, $t=-3$, $p=-3$, $q=1$.

$$\therefore p=\sqrt{\frac{(34)^2+(6)^2+(26)^2}{14}}=\tfrac{1}{7}\sqrt{(6538)}=11.55.$$

Also solved by Mary R. Beck, A. H. Holmes, and J. Scheffer, and the Proposer.

307. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

A family of planes containing a common line intersects a sphere. Find the orthogonal trajectories of the traces. An analytic solution is preferred.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let the sphere and family of planes be respectively,

$$x^2+y^2+z^2-2az=0,\quad z+c=\lambda x\ldots(1),$$

in which λ is the parameter of the planes, so that the common line is

$$z+c=x=0\ldots(2).$$

The following equations define an inversion in space,

$$x=\frac{2a^2x'}{x'^2+y'^2+z'^2},\quad y=\frac{2a^2y'}{x'^2+y'^2+z'^2},\quad z=\frac{2a^2z'}{x'^2+y'^2+z'^2}\ldots(3).$$

The result of (3) applied to (1) is, after reduction,

$$z'-a=0,\quad c(x'^2+y'^2)+ca^2+2a^3-2a^2\lambda x'=0\ldots(4).$$

Thus the circles defined by (1) have become the circles in (4), all lying in one plane. The family of circles orthogonal to (4) is

$$z'-a=0,\quad c(x'^2+y'^2)-ca^2-2a^3-2a^2\mu y'=0\ldots(5),$$

in which μ is the parameter.

Solving (3) for x', y', z', gives formulas for inverting (4) and (5). The former, of course, becomes (1) again, while the latter becomes

$$x^2+y^2+z^2-2az=0,\quad z(a+c)-ac+\mu ay=0\ldots(6).$$

It is a property of inversion that angles remain invariant, so that the system (1) and (6) is orthogonal and (6) is the desired trajectory, composed of circles of which the respective planes have the common line

$$z(a+c)-ac=y=0\ldots(7).$$

The inversion of (3) was so chosen that the original sphere became a plane, thus making the solution depend upon the simpler problem of finding the orthogonal trajectory of a family of plane curves.

Also solved by G. B. M. Zerr.

CALCULUS.

81. **Proposed by J. OWEN MAHONEY, M. Sc., Dallas, Texas.**

Solve, $y^2\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}=bx.$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $y=A+Bx+Cx^2+Dx^3+Ex^4+\dots$

$dy/dx=B+2Cx+3Dx^2+4Ex^3+55Fx^4+\dots$

$d^2y/dx^2=2C+6Dx+12Ex^2+20Fx^3+30Gx^4+\dots$

$y^2=A^2+B^2x^2+C^2x^4+2ABx+2ACx^2+2ADx^3+2AEx^4+2BCx^3$ $+2BDx^4+\dots$

$\therefore y^2(d^2y/dx^2)+a(dy/dy)^2=bx$ gives us

$$\begin{array}{r|l|l|l|l|l}
bx=2A^2C & +6A^2D & x+12A^2E & x^2+20A^2F & x^3+30A^2G & x^4+\dots \\
+aB^2 & +4ABC & +\ 2B^2C & +\ 6B^2D & +12B^2E & \\
 & +4aBC & +12ABD & +24ABE & +\ 2C^3 & \\
 & & +\ 4AC^2 & +16ACD & +40ABF & \\
 & & +\ 4aC^2 & +\ 4BC^2 & +28ACE & \\
 & & +\ 6aBD & +\ 8aBE & +12AD^2 & \\
 & & & +12aCD & +16BCD & \\
 & & & & +\ 9aD^2 & \\
 & & & & +10aBF & \\
 & & & & +16aCE &
\end{array}$$

Equating like powers of x we get

$$C=-\frac{aB^2}{2A^2},\quad D=\frac{2A^2b+4aAB^3+4a^2B^3}{12A^4},$$

$$E=\frac{aB^4[A^2-(A+a)\ (4A+3a)]-abA^2B-2bA^3B}{12A^6}.$$

$$\therefore y=A+Bx-\frac{aB^2}{2A^2}x^2+\frac{2A^2b+4AB^3+4a^2B^3}{12A^4}x^3$$

$$+\frac{aB^4[A^2-(A+a)(4A+3a)-abA^2B-2bA^3B}{12A^6}x^4+\dots$$

where A and B are constants of integration.

This solution does not give a unique result.

233. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin nx}{\sin x}dx$, and (b) $\int_0^{\frac{1}{2}\pi}\frac{\sin^2 nx}{\sin x}dx$, where n is a positive integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. SCHEFFER, A. M., Hagerstown, Md.

$$(a)\ \frac{\sin nx}{\sin x}=2[\cos x+\cos 3x+\cos 5x+\ldots+\cos(n-1)x],\ n \text{ even},$$

$$=1+2[\cos 2x+\cos 4x+\cos 6x+\ldots+\cos(n-1)x],\ n \text{ odd}.$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin nx}{\sin x}dx=2\left(\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\ldots+\frac{1}{n-1}\sin(n-1)x\right)_0^{\frac{1}{2}\pi}$$

$$=2\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\ldots\pm\frac{1}{n-1}\right),\ n \text{ even}=\tfrac{1}{2}\pi \text{ if } n=\infty.$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin nx}{\sin x}dx=\left(x+\sin 2x+\tfrac{1}{2}\sin 4x+\tfrac{1}{3}\sin 6x+\ldots\frac{2}{n-1}\sin(n-1)x\right)_0^{\frac{1}{2}\pi}$$

$$=\tfrac{1}{2}\pi \text{ when } n \text{ is odd}.$$

$$(b)\ \frac{\sin^2 nx}{\sin x}=\sin x+\sin 3x+\sin 5x+\ldots+\sin(2n-1)x.$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2 nx}{\sin x}dx=-\left(\cos x+\tfrac{1}{3}\cos 3x+\tfrac{1}{5}\cos 5x+\ldots+\frac{1}{2n-1}\cos(2n-1)x\right)_0^{\frac{1}{2}\pi}$$

$$=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\ldots+\frac{1}{2n-1}.$$

Also solved by Prof. F. Anderegg.

DIOPHANTINE ANALYSIS.

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139. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

$2^{n-1}(2^n-1)$ is a multiply perfect number of multiplicity 2 when 2^n-1 is prime. Prove that there are no other multiply perfect numbers containing only 2 distinct primes.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $2^n-1=b$.

Then $\dfrac{2^n-1}{2^{n-1}(2-1)}\cdot\dfrac{b^2-1}{b(b-1)}=\dfrac{2^n-1}{2^{n-1}}\cdot\dfrac{b+1}{b}=\dfrac{2^n-1}{2^{n-1}}\cdot\dfrac{2^n}{2^n-1}=2.$

$\therefore 2^{n-1}(2^n-1)$ is a multiply perfect number of multiplicity 2. The second part of the problem is demonstrated in problem 137, Vol. XIII, Nos. 8-9.

140. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Determine (any way) whether the Diophantine equation $\left(\frac{2x-1}{3}\right)=x^2+y^2$ has any positive integer solutions.

Solution by JACOB WESTLUND, Ph. D., Purdue University.

In order that $\frac{2x-1}{3}$ shall be an integer we must have $x=2+3a$, where a is a positive integer. Hence $(1+2a)^3=(2+3a)^2+y^2$ or after a few reductions $y^2=8a^3-6a+3(a^2-1)$.

If a is odd, this equation is impossible, since in that case y must be even and hence all the terms except $6a$ divisible by 4.

If a is even, we put the equation in the form $y^2=8a^3+3a(a-2)-3$. This shows that y must be odd and $y\ +3$ divisible by 8. Hence, setting $y=2b+1$, $4b^2+4b+4$ should be divisible by 8 or $b(b+1)+1$ divisible by 2, which is impossible. Hence the given equation has no positive integer solutions.

Also solved by A. H. Holmes.

No solution of 141 has yet been received.

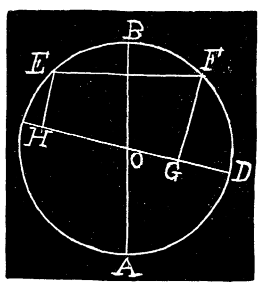
AVERAGE AND PROBABILITY.

178. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random planes cut a given sphere. What is the chance that they intersect within the sphere?

I. Solution by HENRY HEATON, Belfield, N. D.

Let AB and CD be axes of the sphere perpendicular to the two planes and let EF be a trace of one of the planes.



Put $x=OI$, the distance of the plane through EF from the center of the sphere. Put $\theta=\angle BOC$. Then HG, the projection of EF upon $CD=2\sqrt{(a^2-x\ )}\sin\theta$.

It seems to be generally understood that the number of directions of the plane perpendicular to CD depends upon the number of different directions possible to CD, and that this depends upon the number of points in the surface of the sphere. Hence the number of planes of the direction θ is proportional to $\sin\theta$. The angle θ being supposed fixed the chance of intersection within the sphere is $\frac{HG}{CD}=\frac{\sqrt{(a^2-x^2)}\sin\theta}{a}$.

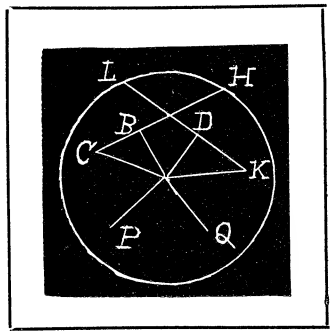
Hence the required probability is

$$p=\int_0^{\frac{1}{2}\pi}\int_0^a\sqrt{(a^2-x^2)}\sin^2\theta\, d\theta\, dx \Big/ \int_0^{\frac{1}{2}\pi}\int_0^a a\sin\theta\, d\theta\, dx$$

$$=\frac{\pi}{4}\int_0^{\frac{1}{2}\pi}\sin^2\theta\, d\theta \Big/ \int_0^{\frac{1}{2}\pi}\sin\theta\, d\theta=\frac{\pi^2}{16}.$$

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Let CH, LK be the diameters of the sections of the sphere made by the planes. B, D their centers; O the center of the sphere; OQ a line such that a line in the plane LK is parallel to the plane DOQ. $OC=OK=r$, $\angle COB=\theta$, $\angle KOD=\phi$, $\angle DOQ=\psi$. The limits of θ and 0 are $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$; of ψ $\pm(\theta-\phi)$ and $\theta+\phi$. The double sign is used $+$ for $\theta>\phi$, $-$ for $\theta<\phi$. Hence the chance p is



$$p=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} d\theta\, d\phi\, d\psi \Big/ \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi} d\theta\, d\phi\, d\psi$$

$$=\frac{4}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\pm(\theta-\phi)}^{\theta+\phi} d\theta\, d\phi\, d\psi=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\phi\, d\theta\, d\phi=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} d\theta=\tfrac{1}{2}.$$

NOTE.—These two solutions differ in the method of distributing the direction of the random planes. ED. F.

NOTES AND NEWS.

Professor C. Alasia, mathematical editor of the "Rivista di Fisica e Matematica," of Pisa and Pavia, will review in that journal all new publications sent to him at Ozieri, Italy.

At the University of Chicago, Assistant Professor L. E. Dickson has been promoted to an associate professorship in mathematics, and Associate Professor Heinrich Maschke to a full professorship in mathematics.

The following courses in Mathematics and Mathematical Astronomy are to be given at the University of Chicago during the Summer Quarter of 1907 beginning June 15th: By Professor Moore: Graphical Methods in Algebra especially for teachers, 4 hours; Theory of Determinants, Advanced Course, 4 hours; General Seminar, 2 hours. By Professor Bolza: Theory of Functions of Complex Variables, 4 hours; Problems in Theory of Functions, 2 hours; Abelian Functions, 2 hours. By Assistant Professor Slaught: Integral Calculus, 5 hours; Differential Equations, 4 hours. By Associate Professor Dickson: Trigonometry, 5 hours; Solid Analytical Geometry, 5 hours; Continuous Groups, 4 hours. By Assistant Professor Moulton: Descriptive Astronomy, 5 hours; Introduction to Celestial Mechanics, 4 hours. By Assistant Professor Laves: Descriptive Astronomy, 5 hours; General Astronomy and Observatory Work, 5 hours. By Dr. Lunn: Curve Tracing and Differential Calculus, 5 hours; Dynamics of Oscillatory Systems, 4 hours. By Mr. Lennes: Plane Analytic Geometry, 5 hours; Critical Review of Secondary Mathematics, 4 hours. In the College of Education: By Professor Myers: Pedagogy of Elementary School Mathematics, Pedagogy of Secondary School Mathematics.

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CO-POLAR AND CO-AXIAL TRIANGLES IN CONICS.

By REV. ALAN S. HAWKESWORTH, Pittsburg, Penn.

Theorem 1. If an inscribed and a circumscribed triangle to any conic curve touch in the same three points, then will they be co-polar in a point within the curve; and co-axial in the polar of said point [Figs. 1 and 2].

Draw the tangents $PR'Q'$, $R'QP'$, and $RP'Q'$, of any three points PQR upon any conic curve, meeting each other, and forming the circumscribing triangle $P'Q'R'$. And join PQR, forming an inscribed triangle, which touches the conic in the same three points as $P'Q'R'$. Draw Cp, Cq, and Cr, the semi-diameters in ellipse [Fig. 1], hyperbola, or circle, parallel to the tangents at P, Q, and R, respectively. Or in the parabola, the lines from any point C to the axis, or any diameter, parallel to said tangents.

Then, by a well known theorem, $PR'=R'Q=Cp:Cq$, and $RQ'=PQ'=Cr:Cp$, and $QP'=P'R=Cq:Cr$. So that, multiplying and cancelling, $PR'.RQ'.QP'=R'Q.PQ'.P'R$. And therefore, by a known theorem in "Transversals," lines PP', QQ', and RR' must concur in a point O, or the inscribed and circumscribed triangles PQR, and $P'Q'R'$ be co-polar in O.

But as so co-polar, they must also be co-axial, by a known theorem, a fact also evident from "Poles and Polars." For if chords PQ and RQ cut the tangent of R and P in J and I, respectivly, while line JI and chord PR concur in H, and HQ be joined, then will the three summits of quadrangle $PIJR$ be H, Q', and Q on the curve. So that $Q'Q$ is the polar of H; and HQ thus the tangent at Q, identical with $R'QP'$.

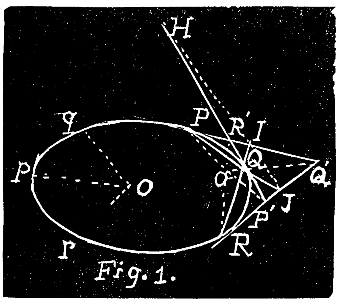


Fig. 1.

Next, HIJ, the co-axial line of triangles PQR and $P'Q'R'$, is also the polar, in respect to the conic, of O, their co-polar point. For $RP'JQ'$ being a harmonic range of quadrangle $PIJR$; PR, POP', PQJ and PIQ' is also a harmonic pencil; and POP' thus the polar of I, and its chord of tangential contact. Similarly, $PR'IQ'$ being a harmonic range, gives us

RP, ROR', RQI and RJQ' as a harmonic pencil; and ROR' thus as the polar of J, while $Q'QO$ is the polar of H. So that O, the common point of POP', ROR' and $Q'QO$, must be the pole to polar HIJ, in respect to the conic.

And lastly, the co-polar point O ever falls within the curve. For when an ellipse, circle, parabola, or one branch of a hyperbola is in question, then the co-polar point O must obviously fall within the inscribed triangle PQR, and hence also within the curve, while, when points PQ, say, lie on one branch, and R on the opposite branch of a hyperbola [Fig. 2], then the co-axial line HIJ plainly cuts internally, the two triangles PQR and $P'Q'R'$, and hence must lie between the branches, so that its pole O still lies within the curve, and within that branch upon which lies the base PR of the inscribed triangle, but now beyond said base and between the produced sides PQ and RQ.

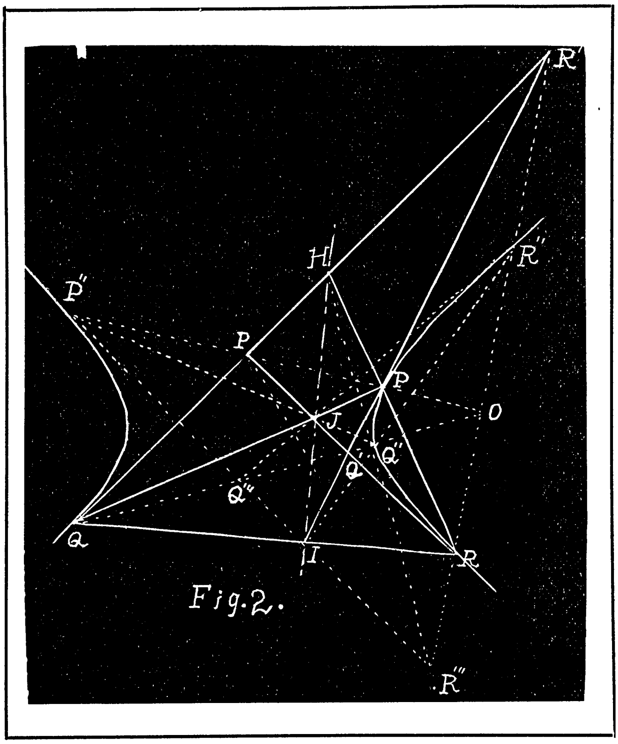


Fig. 2.

Theorem 2. If there be an inscribed and a circumscribed triangle to any conic curve, touching it in the same three points, and thus co-polar in a point O, within the curve, and co-axial along the polar of said point, and if the three radiant co-polar points cut the conic again in three fresh points and these points be joined, and their tangents be also drawn, then the resultant four triangles—two inscribed, and two circumscribed—will be commonly co-polar in the radiant point O within the curve; and also will be, not merely commonly co-axial along its polar, but also co-axial in the same three fixed points HIJ.

Take any conic—say both branches of the hyperbola [Fig. 2]—and let PQR and $P'Q'R'$ be an inscribed and circumscribed triangle, touching it in the same three points P, Q, and R, and hence co-polar in O within the curve, and co-axial in HIJ along its polar [Theorem 1]. Let the three radiant lines $OPP'P''$, $OQ''Q'Q$, and $ROR''R'$ cut the conic again in points P'', Q'', and R'', respectively. Join these three fresh points, and draw $P''Q'''R'''$, $Q''P'''R'''$, and $R''P'''Q'''$, their tangents, cutting in Q''', R''', and

P''', respectively, giving us, therefore, two inscribed triangles, PQR and $P''Q''R''$, and two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$. We will now show that O is the common co-polar point of all four triangles, and HJI, along its polar, their three common collinear co-axial points.

For, by construction, O is the co-polar point of the inscribed triangles PQR and $P''Q''R''$. And if their sides PR and $P''R''$ cut in, say h, while QR and $Q''R''$ cut in, say i, and PQ, $P''Q''$ cut in, say j; then will O and h be summits of quadrangle $PRR''P''$, and likewise Oi summits of quadrangle $QRR''Q''$, and Oj summits of quadrangle $PQQ''P''$. Hence hji is the polar of O with respect to the conic; thus coinciding with line HJI. But the three points where the polar of O, the co-axial line of triangles PQR and $P'Q'R$, cuts the sides of triangle PQR, are HJI. So that lines HJI and hji not only coincide; but points Hh, Jj, and Ii, respectively, are identical; so that HJI are the common collinear co-axial points of the three co-polar triangles PQR, $P'Q'R'$, and $P''Q''R''$; their corresponding sides PR, $P'R'$, and $P''R''$ concurring in H; sides PQ, $P'Q'$, and $P''Q''$ concuring in J, and QR, $Q'R'$, and $Q''R''$ in I.

Then lastly, H being a summit of the quadrangle $PJIR$, must be the pole to the polar $QQ'Q''O$ joining its other two summits Q' and Q, and hence in it meet the two tangents $QP'HR'$ and $R'''Q''P'''H$, while the quadrangle $PJIR$ again gives us $RQ'JP'$ as a harmonic range. And thus $PQ'I$ and $OPP'P''$ as conjugate rays to the pencil $PQ'I$, PJQ, $PP'P''$, and RPH. So that in I concur the tangents $R'PQ'I$ and $P''Q'''IR'''$. While similarly, $RQ'JP'$ and $RR''R'$ being conjugate rays in the pencil RQ, RP', RH, and RR', it follows that in J concur the tangents $RQ'J$, and $R''P'''JQ'''$.

So that HJI are the three collinear points in which the two inscribed triangles PQR and $P''Q''R''$, and the two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$ are commonly co-axial, while by Theorem 1, triangles $P''Q''R''$ and $P'''Q'''R'''$, being so co-axial in HJI, must be co-polar in O, its pole, and thus all four triangles be commonly co-polar in said point O. So that $OPP'''P'P''$ are collinear; as also $OQ''Q'Q'''Q$, and $R'''ROR''R'$.

NOTE ON THE QUARTIC.

By DR. R. P. STEPHENS, Wesleyan University, Middletown, Conn.

The general quartic equation

(1) $$ax^4+4bx^3+6cx^2+4dx+e=0,$$

where the coefficients may be real or complex, can be reduced to the form

(2) $$x^4-2\mu x^2+\nu^2=0.$$

The roots of (2) are

$$x = \pm\sqrt{[\mu \pm \sqrt{(\mu^2-\nu^2)}]} \text{ or } x = \pm\frac{1}{\sqrt{2}}[\sqrt{(\mu-\nu)} \pm \sqrt{(\mu+\nu)}].$$

If we make the substitution

$$\sqrt{2}\, t \equiv \sqrt{(\mu-\nu)} \pm (\mu+\nu),$$

then the four roots may be expressed as

$$x_1 = t, \quad x_2 = -t, \quad x_3 = \nu/t, \quad x_4 = -\nu/t.$$

That any quartic which has no repeated roots can be reduced to the form (2) is easily seen geometrically. In general, the roots of a quartic are represented by four points in the complex plane, say x_1, x_2, x_3, x_4. If these be divided into pairs x_1, x_2 and x_3, x_4, there is a pair α, β which is harmonic* with respect to both. That bilinear transformation which throws α into zero and β into infinity will obviously throw x_1 and x_2 into points symmetrical with respect to the origin, and will arrange x_3 and x_4 in a similar manner; and the transformed points will be as given above.

The reduction of the quartic to this simple form is secured analytically as follows: If x_1, x_2, x_3, x_4 are the four roots of (1), then the pair α, β will be harmonic with respect to the two pairs x_1, x_2 and x_3, x_4, provided

$$2\alpha\beta - (\alpha+\beta)(x_1+x_2) + 2x_1x_2 = 0,$$

$$\text{and } 2\alpha\beta - (\alpha+\beta)(x_3+x_4) + 2x_3x_4 = 0;$$

from which

$$\alpha+\beta = \frac{2(x_1x_2 - x_3x_4)}{x_1x_2(x_3+x_4) - x_3x_4(x_1+x_2)}.$$

But since the four roots may be grouped in two other ways, we shall have two other values for $(\alpha+\beta)$. Forming the equation of which these three values of $\alpha+\beta$ are roots and substituting the values for symmetric functions of the roots from (1), we obtain†

$$(3) \qquad 2(a^2d - 3abc + 2b^3)\lambda^3 + (a^2e + 6b^2c - 9ac^2 + 2abd)\lambda^2$$
$$+ 2(abc + 2b^2d - 3acd)\lambda + b^2e - ad^2 = 0,$$

where $\lambda \equiv \alpha+\beta$.

In a similar way, from the reciprocals of the roots, we obtain

$$(4) \qquad 2(be^2 - 3cde + 2d^3)\lambda'^3 + (ae^2 + 6cd^2 - 9c^2e + 2bde)\lambda'^2$$
$$+ 2(ade + 2bd^2 - 2bce)\lambda' + ad^2 - b^2e = 0,$$

*For this general use of harmonic pairs, see Harkness and Morley, *Introduction to Analytic Functions*, pp. 32—34.

†See Burnside and Panton, *Theory of Equations*, p. 130 (3rd Ed.).

where $\lambda' \equiv 1/\alpha + 1/\beta$. From λ and λ', we can obtain α and β.

If now α_1, β_1 are a harmonic pair obtained from (3) and (4), then the bilinear transformation

$$x = \frac{\beta_1 x' + \alpha_1}{x' + 1}$$

will reduce the general quartic (1) to the required form.

In case two of the roots of (1) are equal, say $x_1 = x_2$, then equation (3) will have a repeated root which will equal twice the reciprocal of x_1, the repeated root of (1); and if three roots of (1) are equal, then all harmonic pairs coincide at the repeated root and (3) will be a perfect cube. Thus we see that the general quartic with real or complex coefficients can be reduced to the form (2) in this way.

REMARKS ON THE REPORT ON GEOMETRY OF THE COMMITTEE* OF THE CENTRAL ASSOCIATION.

By PROFESSOR GEORGE BRUCE HALSTED, Greeley, Colorado.

This report is epoch-making,—at the very least epoch-marking.

There simply *must* be things not *explicitly* defined, and point, straight, plane, between, should be among them. The Report says: "It is recommended that the term *sect* be used for segment of a straight line lying between two of its points." In fact the term "sect" has 'arrived.' The Encyclopaedia Americana uses it. This Report uses it more than twenty-two times. It is used not less than twenty-four times in the remarkable Presidential Address by Professor Alfred Baker of the University of Toronto to the Royal Society of Canada.

Think of the neatness with which the bunglesome phrase "transferrer of straight-line segments" becomes *sect-carrier.* Realize how elegantly "the algebra or algorithm of straight-line segments" becomes *sect-calculus.*

"Instead of *axioms,*" says the Report, "use geometrical *assumptions.*" It gives for example Pasch's assumption, now so renowned. From the list of assumptions we mention: "3. A point on a straight line divides it into two parts, called *rays.*" "6. A sect has one and only one mid point." "11. A straight line divides the points not on it into two classes such that sects determined by two points of the same class are not intersected by the line, and sects determined by two points not of the same class are intersected by the line."

*The Committee consist of G. W. Greenwood, Chairman, Salem, Va.; C. A. Pettersen, Chicago, Ill.; C. E. Comstock, Peoria, Ill., and C. W. Newhall, Faribault, Minn. Copies of the report may be had by sending a stamp to Miss Mabel Syker, 438 East 57th Street, Chicago, Ill. ED. S.

"Definitions," says the Report, "should not be based upon crude images, affording little upon which reasoning may lay hold. For example, 'An angle is the opening between two lines which meet.' Instead, define an angle as the figure formed by two rays having a common origin." A demonstration in which we use information obtained by looking at a figure is not of the highest order.

The one serious slip in the Report is the sentence: "Also, in 'A line perpendicular to each of two intersecting lines (at their intersection) is perpendicular to their plane,' we assume that two intersecting lines have a common perpendicular though we cannot justify the assumption by any previous proposition." Halsted's Rational Geometry here makes no assumption whatever. Its figure for this proposition is already covered by the preceding proposition: On any straight to put two planes; and the problem: To erect a perpendicular to a straight from any point on it.

We must agree with the Report, that the treatment of mensuration in most texts is extremely unfortunate. In fact measurement in terms of a common unit at once introduces incommensurability and irrational numbers. No geometry exists in which irrational numbers are adequately treated. Halsted's Rational Geometry outwits the difficulty.

The committee recommends that a critical course in elementary geometry be offered in courses of study in colleges.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Remarks on Two Solutions of Problem 89. By G. A. MILLER.

The following solutions present very instructive examples of fallacious reasoning and arriving at the answer by a remarkable coincidence. As the problem is so well known, and these solutions are said to have appeared in other scientific journals it seems desirable to enter into some details. The problem is as follows:

Solve by quadratics, $x^2+y=7\ldots(1)$, $x+y^2=11\ldots(2)$.

We shall first speak of the solution given on page 37, Volume VI, of this journal. To make the matter as clear as possible we shall employ the language of analytic geometry. The problem is to find the common points (or at least one of them) of two intersecting parabolas. The author of the solution in question subtracts (2) from (1), and thus obtains the equation of an equilateral hyperbola containing the four common points of the given parabolas. The equation of this hyperbola is

$$y^2-x^2-(y-x)=4\ldots(3).$$

Substituting a for $x+y$ and b for $y-x$ tnis equation reduces to

$$ab-b=4\ldots(4).$$

While hyperbola (4) is satisfied by an infinite number of pairs of values for a and b, the author arrives at the pair $a=5$, $b=1$ by taking the following steps.

From (4) we may obtain by transposing and squaring,

$$a^2b^2=16+8b+b^2\ldots(5),$$

and also
$$-10ab+25=-15-10b\ldots(6).$$

Adding (5) and (6) there results

$$(ab-5)^2=(1-b)^2,$$

or
$$ab-5=1-b,\ ab+b=6\ldots(7).$$

Combining (4) and (7), it follows that $b=1$ and $a=5$.

The author has thus found one point on hyperbola (4), and the remarkable coincidence is that this is a point of intersection of the parabolas (1) and (2). Hence $x+y=5$, $y-x=1$ lead to $x=2$, $y=3$, which is a solution of the equations. That the method is erroneous follows directly from the fact that the same auxiliary equations could be obtained from

$$x^2+y=k\ldots(1),$$
$$x+y^2=k+4\ldots(2),$$

where k is arbitrary. If we make $k=0$, for instance, it is evident that $x=2$, $y=3$ does not satisfy the system. The solution is a good illustration of blindly manipulating algebraic expressions. Equations (3) and (4) represent a hyperbola which goes through the points whose co-ordinates are desired, and the rest of the solution is merely a very laborious method of finding the co-ordinates of one point on this hyperbola. As the number of points on the hyperbola is infinite, while only four of these points are common to the given parabolas, the probability that this method should lead to a correct result in a given problem is zero, and yet it happened to do so in the present instance.

The second solution of the same problem, to which we desire to call attention, is published on page 13 of the same volume. Equations (1) and (2) are written as follows:

$$x^2-9=2-y=d, \text{ by assumption},$$
$$x-3=4-y^2=sd, \text{ by assumption}.$$

Hence
$$x^2-9=(x-3)/s=x/s-3/s.$$

Completing the square we have

$$x^2-x/3+1/4s^2=9-3/s+1/4s^2.$$

Hence $$x-1/2s=3-1/2s,\text{ or } x=3.$$

The fallacy will at once appear if it is observed that the same arguments could be used with respect to the two equations

$$x^2+y=9+\alpha\ldots(1),$$
$$y^2+x=3+\beta\ldots(2).$$

These may be written as follows:

$$x^2-9=\alpha-y=d\ldots(3),$$
$$x-3=\beta-y^2=sd\ldots(4).$$

As x cannot be equal to 3 for an arbitrary pair of values of α, β it follows that the method is fallacious. Just as in the preceding solution it is implicitly assumed that any point on $x^2-9=(x-2)/s$ must be common to the given parabolas. And the probability that this method should lead to the correct result in a given problem is again zero.

278. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$xyz(\Sigma x)^2<3\Sigma y^2z\,\Sigma yz^2$, if x, y, z are positive.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{x^2y+y^2z+z^2x}{3}>(x^2y.y^2z.z^2x)^{1/3}>xyz.\quad\text{Also } \frac{xy^2+yz^2+zx^2}{3}>xyz.$$

$\therefore 3\,\Sigma y^2z\,\Sigma yz^2>27x^2y^2z^2>xyz.27xyz.$
$(x+y+z)^2$ is the greatest when $x=y=z$, and is then equal to $9x^2$.
$\therefore (x+y+z)^2<27xyz.\quad \therefore xyz[\Sigma(x)]^2<3\,\Sigma y^2z\,\Sigma yz^2.$

279. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

The United States Panama Canal Bonds were issued, to date August 1, 1906, and will mature on August 1, 1936; and they bear interest at the rate of 2% per annum, payable quarterly, on the first day of November, 1906, and the first day of February, May, and August, 1907, and so on for each succeeding quarter, until the bonds mature, when the principal will be paid at par with the last quarter's interest. The coupon bonds of this loan were quoted on the New York Stock Exchange, at 10.30 a. m., on December 17, 1906, at 103¾ bid and 104¾ asked.

Required: The rate of interest per annum, payable quarterly, an investor would *realize* if he purchased the Panama bonds on December 17, 1906, and could reinvest his interest income, quarterly, at the *realized* rate.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let P=price of bond, n=number of quarter years to run, S=face of bond, x=realized rate of interest, r=rate of interest bond bears.

$P(1+\frac{1}{4}x)^n$=value of purchase money at the end of n quarter years.

$$\frac{Sr}{4}(1+\tfrac{1}{4}x)^{n-1}+\frac{Sr}{4}(1+\tfrac{1}{4}x)^{n-2}+\frac{Sr}{4}(1+\tfrac{1}{4}x)^{n-3}+\ldots+Sr+S$$

$$=S+\frac{\frac{Sr}{4}[(1+\frac{1}{4}x)^n-1]}{\frac{1}{4}x}$$

is the amount of money received on the bond.

$$\therefore P\tfrac{1}{4}x(1+\tfrac{1}{4}x)^n=\frac{Sx}{4}+\frac{Sr}{4}(1+\tfrac{1}{4}x)^n-\frac{Sr}{4}.$$

$$\therefore 104\tfrac{3}{4}(1+\tfrac{1}{4}x)^{119}x=100x+2(1+\tfrac{1}{4}x)^{119}-2,$$

$$(104\tfrac{3}{4}x-2)(1+\tfrac{1}{4}x)^{119}=100x-2,$$

$$\therefore x=.01794=1.794\%.$$

II. Solution by the PROPOSER.

The solution of the above problem requires that certain technical considerations be observed. The bond runs for 30 years, and as the interest is payable quarterly there are attached to it 120 coupons of \$0.50 each. The exact number of days in each quarter must be noted. The interest for the first quarter was paid November first. The next quarter (November, December, and January), contains 92 days, and interest has accrued from November first to include December sixteenth, or for 46 days; or the accrued interest is equal to $\frac{46}{92}$ of a quarter; or is equal to $\frac{1}{2}$ of \$0.50, or \$0.25, to the date of purchase. There is left to the date of the maturity $120-1.5$ or 118.5 quarters. The lowest price bid was \$103.75, and highest price asked was \$104.75. If there was a sale of these bonds on December 17 it is fair to assume that it was at either limit or between those limits, and we therefore assume that the sale was at the mean price or at \$104.25. To find the net investment we subtract the accrued interest from the mean price and have the net investment, or \$104.25—\$0.25, or \$104, the net investment.

Let $4X$=the realized rate of interest per annum, payable quarterly; or let X=the quarterly rate; let P=\$100=the face of the bond; n=the number of interest periods; I=the quarterly interest on \$100; and N=\$104=the net investment.

The amount of the net investment from the date of purchase to the date of maturity would be expressed symbolically by $N(1+X)^n$. The investor will at the date of maturity receive the face of the bond, P=\$100, and the amount of his quarterly interest considered as an annuity compounded at the rate X for the time n, which would be expressed, symbolically, by $I[(1+X)^n-1]/X$.

We can now equate the equivalent terms and obtain the following general equation:

$$N(1+X)^n = P + I[(1+X)^n - 1]/X \ldots (1).$$

By transformation and reduction (1) takes the following general form:

$$N = I/X - (I/X - P)/(1+X)^n \ldots (2).$$

Substituting numerical values we have:

$$\$104 = \$0.50/X - [\$0.50/X - \$100]/(1+X)^{118.5} \ldots (3).$$

We now have to find such a value of X that if it be substituted in (3) the two members will be equal. As the bond is sold at a premium we know that X must be less than 0.005. By trial we find it to be greater than 0.0045. We will first try 0.00456, and then 0.00457.

Take $X=0.00456$ in (2) and (3) and we have $I/X = \$0.50/0.00456 = \$109.649123$; from this amount take \$100 and we have \$9.649123. The logarithm of 9.649123 is 0.9844878. From this logarithm take 118.5 times the logarithm of 1.00456, or 0.2341442, and we have the logarithm 0.7503436; which is the logarithm of the number 5.627864; this number taken from 109.649123 gives 104.021259. This number is greater than the number in the first member of equation (3), or than 104, by 0.021259, which shows that the assumed value of X, or 0.00456, is too small.

Take $X=0.00457$ in (2) and (3) and we have $I/X = \$0.50/0.00457 = \$109.40919$; from this amount take \$100, and we have \$9.40919. The logarithm of 9.40919 is 0.9735522; from this logarithm take 118.5 times the logarithm of 1.00457, or 0.2346537, and we have the logarithm 0.7388985, which is the logarithm of the number 5.47149; this number taken from the number 109.40919 gives 103.93770. This number is less than the number in the first member of equation (3), or than 104, by 0.0623, which shows that the assumed value of X, or 0.00457, is too large.

Now as the assumed value, $X=0.00456$, gives a value too small, and the assumed value, $X=0.00457$, gives a value too large, the true value of X lies between these two assumed rates, the first assumption giving the nearer value. The approximate increase may be determined by Double Position:

$$\begin{array}{ccccccc} 104.02126 & & 0.00457 & & 104.021259 & & \\ 103.93770 & & 0.00456 & & 104.000000 & & \\ \hline 0.08556 & : & 0.00001 & :: & 0.021259 & : & \text{error} = e. \end{array}$$

Reducing the proportion we find $e=0.00000254$; add this to 0.00456 and we have 0.00456254 as a new trial value for X.

Try $X=0.0045625$ in equation (3) and we have a result approximately within a few mills of a true value.

Therefore the rate of interest per quarter which the investor will realize is $X=0.45625\%$; or $4X=1.825\%$, the investor's rate per annum, payable quarterly.

When there are three rates of interest to be considered in the problem, or the nominal rate expressed in the bond, the investor's rate, and the current or market rate, each per annum, payable quarterly, we let $4Y$=the current rate per annum, payable quarterly, then Y=the current quarterly rate. Put this value in equation (1) and we have:

$$N(1+X)^n=P+I[(1+Y)^n-1]/Y\ldots(4).$$

With equation (4) we have a perfectly general equation from which, by proper substitutions, we may apply it to any loan and find all values required, as N, P, I, n, X, and Y.

The logarithmic calculations in the above solution were made with Shortrede's tables of logarithms and antilogarithms. Should it be necessary to carry the decimals further it would be better to use ten-place tables.

Dr. Zerr's result differs from Mr. DeLand's in that he disregarded the technical consideration required in these bonds. The problem was also solved by A. H. Holmes. ED. F.

280 (Incorrectly numbered 180). Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find values of x, y, z, and u satisfying the equations

$$x+y+z+u=10\ldots[1],$$
$$x^2+y^2+z^2+u^2=30\ldots[2],$$
$$x^3+y^3+z^3+u^3=100\ldots[3],$$
$$x^4+y^4+z^4+u^4=354\ldots[4].$$

Solution by E. A. ECKHARDT, 903 North Fifth Street, Philadelphia, Pa.

By multiplication, addition, and subtraction we obtain

$\Sigma xy=35$
$\Sigma x^2y=300$
$\Sigma xyz=50$
$\Sigma xy^3=646$
$\Sigma x^2y^2=273$
$\Sigma x^2yz=404$
$xyzu=24$

$$\begin{aligned}\Sigma xy^3&=x^3y+x^3z+x^3u+y^3x+y^3z+y^3u+z^3x+z^3y\\&\quad+z^3u+u^3x+u^3y+u^3z.\\&=x^2(xy+xz+xu+yz+yu+zu)\\&\quad-x(xzu+xyu+xyz+yzu)+xyzu\\&\quad+y^3(x+u+z)+z^3(x+y+u)+u^3(x+y+z)\\&=x^2\Sigma xy-4\Sigma xyz+xyzu+y^3(10-y)\\&\quad+z^3(10-z)+u^3(10-u).\end{aligned}$$

$$\Sigma xy^3=x^2\Sigma xy-4\Sigma xyz+xyzu+10(y^3+z^3+u^3)-(y^4+z^4+u^4).$$

Substituting, we have $646=35x^2-50x+24+10(100-x^3)-(354-x^4)$.

Whence, we have $x^4-10x^3+35x^2-50x=-24\ldots(2)$,

or $x^4-10x^3+35x^2-50x+24=0$.

The equations in y, z, and u are found to be identical to (2). The

roots of (2) are easily found to be $x=1$, $x=2$, $x=3$, $x=4$. We have, therefore, the following four groups of values:

$$\begin{cases} x=1, & y=2, & z=3, & u=4; \\ x=2, & y=1, & z=4, & u=3; \\ x=3, & y=4, & z=1, & u=2; \\ x=4, & y=3, & z=2, & u=1. \end{cases}$$

Also solved by G. B. M. Zerr, J. Scheffer, and A. H. Holmes.

GEOMETRY.

308. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the locus of O, if the differences of the squares of tangents from it to circles A, B, C are x^2, y^2, z^2, respectively.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Maine; L. E. NEWCOMB, Los Gatos, California; and J. SCHEFFER, A. M., Hagerstown, Md.

Let (u, v) be the co-ordinates of the point O;

$(x-m_1)^2+(y-n_1)^2=R_1^2$, the equation of the circle A,

$(x-m_2)^2+(y-n_2)^2=R_2^2$, the equation of the circle B, and

$(x-m_3)^2+(y-n_3)^2=R_3^2$, the equation of the circle C.

Then $(u-m_1)^2+(v-n_1)^2-R_1^2=T_1^2$, the square of the tangent from O to A;

$(u-m_2)^2+(v-n_2)^2-R_2^2=T_2^2$, the square of the tangent from O to B;

$(u-m_3)^2+(x-n_3)^2-R_3^2=T_3^2$, the square of the tangent from O to C.

$\therefore 2(m_2-m_1)u+2(n_2-n_1)v+m_1^2+n_1^2-m_2^2-n_2^2+R_2^2-R_1^2=x^2 \ldots (1)$,

$2(m_3-m_1)u+2(n_3-n_1)v+m_1^2+n_1^2-m_3^2-n_3^2+R_3^2-R_1^2=y^2 \ldots (2)$,

$2(m_3-m_2)u+2(n_3-n_2)v+m_2^2+n_2^2-m_3^2-n_3^2+R_3^2-R_2^2=z^2 \ldots (3)$.

Adding (1) and (3) and subtracting (2), we have $x^2+z^2=y^2$ or $y^2-x^2=z^2$, the former being the equation of a circle with a variable radius y, and the latter the equation of an equilateral hyperbola with a variable semi-axes z.

If x, y, and z are constants, the locus is a point.

309. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

To find the equation of Brocard's Ellipse, the sides b and c of the triangle being the axes of co-ordinates.

Solution by the PROPOSER.

Let AB be the axis of x, AC that of y, then the equation of any ellipse touching the three sides of the triangle is of the form $D^2y^2+4Bxy+E^2x^2+4Dy+4Ex+4=0$, where

$$B = -\frac{8+4bD+4cE+bcDE}{2bc}.$$

The co-ordinates of the center of the ellipse are $-\dfrac{2D}{DE+2B}$ and $-\dfrac{2E}{DE+2B}$, or substituting the value of B, $\dfrac{Dbc}{2(2+bD+cE)}$ and $\dfrac{Ebc}{2(2+bD+cE)}$. The co-ordinates of the two Brocard Points O and O' are, respectively,

$$\frac{a^2b^2c}{a^2b^2+a^2c^2+b^2c^2},\ \frac{b^3c}{a^2b^2+a^2c^2+b^2c^2},\ \text{and}\ \frac{b^2c^3}{a^2b^2+a^2c^2+b^2c^2},\ \frac{a^2c^2b}{a^2b^2+a^2c^2+b^2c^2}.$$

Therefore, the co-ordinates of the middle point of OO', that is, the center of the ellipse, are

$$\tfrac{1}{2}\cdot\frac{b^2c(a^2+c^2)}{a^2b^2+a^2c^2+b^2c^2}\ \text{and}\ \tfrac{1}{2}\cdot\frac{bc^2(a^2+b^2)}{a^2b^2+a^2c^2+b^2c^2};$$

$$\therefore \frac{D}{2+bD+cE}=\frac{b(a^2+c^2)}{a^2b^2+a^2c^2+b^2c^2},\quad \frac{E}{2+bD+cE}=\frac{c(a^2+b^2)}{a^2b^2+a^2c^2+b^2c^2};$$

whence, $D=-\dfrac{2(a^2+c^2)}{bc^2}$, $E=-\dfrac{2(a^2+b^2)}{b^2c}$.

Substituting, we find for the required equation of the Brocard Ellipse,

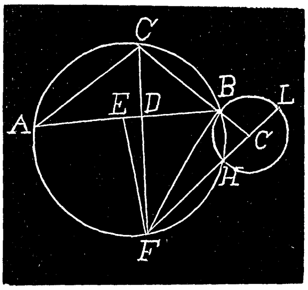
$$b^2(a^2+c^2)^2y^2+2bc[(a^2b^2+a^2c^2+b^2c^2)-a^4]xy+c^2(a^2+b^2)^2x^2$$
$$-2b^3c^2(a^2+c^2)y-2b^2c^3(a^2+b^2)x+b^4c^4=0.$$

Also solved by G. B. M. Zerr, who used trilinear co-ordinates.

310. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

Construct a plane triangle having given the base, the vertical angle, and the bisector of the vertical angle.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; J. SCHEFFER, A. M., Hagerstown, Md., and C. N. SCHMALL, A. B., 89 Columbia Street, New York.



Upon the given base AB construct a circle whose segment ACB shall contain the given vertical angle. Through E, the mid-point of AB, draw EF perpendicular to AB, meeting the circumference at F. Join FB, and perpendicular to FB draw BG equal to one half the given bisector of the vertical angle. With G as center and BG as radius describe the circle BHL, and draw FGL. With F as center, FL as radius, describe a circle cutting the given circle in C. Join FC, cutting AB in D. Then ABC is the triangle required.

In the triangles FCB and FBD, $\angle FCB = \angle FBA$, since arc AF=arc FB; also $\angle CFB$ is common, hence the triangles are similar, and $FC : FB = FB : FD$; but $FL(=FC) : FB = FB : FH$. Therefore $FH = FD$ and $HL = CD$.

Hence in the triangle ABC, AB is the given base, $\angle ACB$ the given vertical angle, and CD the given bisector, and the triangle is satisfied in every condition.

Also solved by L. E. Newcomb, and A. H. Holmes.

CALCULUS.

233. **Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.**

Prove that $\displaystyle\int_0^\infty \frac{x^{a-1}dx}{1+2x\cos\theta+x^2} = \frac{\pi\sin(1-a)\theta}{\sin a\pi\,\sin\theta}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^\infty \frac{a^{-1}dx}{1+2x\cos\theta+x^2} = \frac{1}{a}\int_0^\infty \frac{dx}{\sin^2\theta+(x+\cos\theta)^2}$$

$$= \frac{1}{a\sin\theta}\tan^{-1}\left(\frac{x+\cos\theta}{\sin\theta}\right)_0^\infty = \frac{\theta}{a\sin\theta}.$$

The problem giving the result stated is as follows:

$$\int_0^\infty \frac{x^{a-1}dx}{1+2x\cos\theta+x^2} = \frac{1-\sqrt{(-1)}\cot\theta}{2}\int_0^\infty \frac{x^{a-2}dx}{x+\cos\theta+\sqrt{(-1)}\sin\theta}$$

$$+\frac{1+\sqrt{(-1)}\cot\theta}{2}\int_0^\infty \frac{x^{a-2}dx}{x+\cos\theta-\sqrt{(-1)}\sin\theta} = P.$$

Let $x = y[\cos\theta \pm \sqrt{(-1)}\sin\theta]$, the plus sign for the first term, the minus sign for the second term.

$$\therefore P = \tfrac{1}{2}[1-\sqrt{(-1)}\cot\theta]\,[\cos(a-2)\theta+\sqrt{(-1)}\sin(a-2)\theta]\int_0^\infty \frac{y^{a-2}dy}{1+y}$$

$$+\tfrac{1}{2}[1+\sqrt{(-1)}\cot\theta]\,[\cos(a-2)\theta-\sqrt{(-1)}\sin(a-2)\theta]\int_0^\infty \frac{y^{a-2}dy}{1+y}.$$

$$\therefore P = [\cos(a-2)\theta+\sin(a-2)\theta\cot]\int_0^\infty \frac{y^{a-2}dy}{1+y}$$

$$= \frac{\sin(a-1)\theta}{\sin\theta}\int_0^\infty \frac{y^{a-2}dy}{1+y} = \frac{\sin(a-1)\theta}{\sin\theta}\cdot\frac{\pi}{\sin(a-1)\pi}.$$

$$\therefore P = \frac{\pi\sin(1-a)\theta}{\sin\theta\,\sin a\pi}$$

This problem was incorrectly stated, the error being due to an oversight in reading proof. It is correctly stated above, the numerator being x^{a-1} instead of a^{-1}. ED. F.

234. **Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Find the first negative pedal of an ellipse semi-axes a, b, referred to origin as center, and show that its entire area is $\frac{\pi}{2}\left[\frac{(a^2+b^2)^2}{4ab}+ab\right]$.

Solution by the PROPOSER.

$r\cos(\theta-\phi)=p=\frac{b}{\sqrt{(1-e^2\cos^2\phi)}}$, is the equation of a straight line perpendicular to the radius vector at its extremity. The envelope of this line is the first negative pedal. Differentiating we get

$$\sin(\theta-\phi)\sqrt{(1-e^2\cos^2\theta)}+\frac{e^2\cos(\theta-\phi)\sin\phi\cos\phi}{\sqrt{(1-e^2\cos^2\phi)}}=0.$$

$$\cos(\theta-\phi)=\frac{b}{r\sqrt{(1-e^2\cos^2\phi)}},\quad \sin(\theta-\phi)=\frac{\sqrt{[r^2(1-e^2\cos^2\phi)-b^2]}}{\sqrt{(1-e^2\cos^2\phi)}}.$$

These values from the equation of tho line substituted in the derived equation gives us

$$r^2=\frac{b^2}{(1-e^2\cos^2\phi)^2}-\frac{e^2b^2(1-e^2)\cos^2\phi}{(1-e^2\cos^2\phi)^3}$$

as the pedal sought.

$$\therefore A=2a^2b^2\int_0^{\frac{1}{2}\pi}\left[\frac{a^2}{(a^2\sin^2\phi+b^2\cos^2\phi)^2}-\frac{b^2(a^2-b^2)\cos^2\phi}{(a^2\sin^2\phi+b^2\cos^2\phi)^3}\right]d\theta.$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(a^2\sin^2\phi+b^2\cos^2\phi)^2}=\frac{\pi(a^2+b^2)}{4a^3b^3}.$$

Differentiating this with respect to b we get

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\phi\,d\phi}{(a^2\sin^2\phi+b^2\cos^2\phi)^3}=\frac{\pi(3a^2+b^2)}{16a^3b^5}.$$

$$\therefore A=\frac{\pi a(a^2+b^2)}{2b}-\frac{\pi(a^2-b^2)(3a^2+b^2)}{8ab}=\frac{\pi}{8ab}(a^4+6a^2b^2+b^4)$$

$$=\frac{\pi}{2}\left[\frac{(a^2+b^2)^2}{4ab}+ab\right].$$

Also solved by J. Scheffer.

235. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

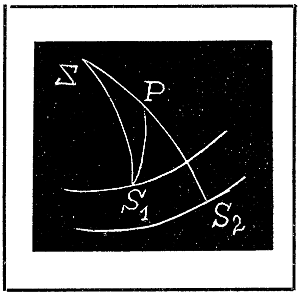
The latitude of a place and two circles parallel to the horizon being given, to determine the declination of a heavenly body whose apparent time of passage from one circle to the other shall be a minimum.

I. Solution by the PROPOSER.

Employing the usual notation, let Z be the zenith, P the celestial pole, S_1, S_2, the positions of a heavenly body on the parallel circles, the polar distances PS_1 and PS_2 being equal.

Let $\angle ZPS_1=\phi_1$, and $\angle ZPS_2=\phi_2$; and let $PS_1=PS_2=x$, arc $ZS_1=\alpha_1$, $ZS_2=\alpha_2$, latitude$=\lambda$, declination$=D$.

Then by the conditions of the problem the heavenly body is to pass along the arc S_1S_2 in the shortest possible time. In other words $\angle S_1PS_2$ is to be a minimum.



$$\therefore \frac{d}{dx}S_1PS_2=\frac{d}{dx}(\phi_2-\phi_1)=0,$$

$$\text{but } \frac{d\phi_1}{dx}=-\frac{\cot S_1}{\sin x} \text{ and } \frac{d\phi_2}{dx}=-\frac{\cot S_2}{\sin x}.$$

$$\therefore \frac{\cot S_1}{\sin x}=\frac{\cot S_2}{\sin x}. \quad \therefore S_1=S_2;$$

$$\text{but } \cos S_1=\frac{\sin\lambda-\cos\alpha_1 \cos x}{\sin\alpha_1 \sin x} \text{ and } \cos S_2=\frac{\sin\lambda-\cos\alpha_2 \cos x}{\sin\alpha_2 \sin x}.$$

$$\therefore \frac{\sin\lambda-\cos\alpha_1 \cos x}{\sin\alpha_1}=\frac{\sin\lambda-\cos\alpha_2 \cos x}{\sin\alpha_2}.$$

$$\text{Whence } \cos x=\frac{\cos\frac{1}{2}(\alpha_2+\alpha_1)}{\cos\frac{1}{2}(\alpha_2-\alpha_1)}.\sin\lambda.$$

The declination being the complement of the polar distance we have, therefore,

$$\sin D=\frac{\cos\frac{1}{2}(\alpha_2+\alpha_1)}{\cos\frac{1}{2}(\alpha_2-\alpha_1)}\sin\lambda.$$

In the special case where $\alpha_1=\frac{1}{2}\pi$, and $\alpha_2=\frac{1}{2}\pi+2\delta$, this expression is reduced to $\sin D=-\tan\delta\sin\lambda$. This formula may have important applications. Thus, take for instance, the case of the sun, and supposing 2δ to be its angular depression below the horizon when twilight begins in the morning or ends in the evening, we can calculate the time of shortest twilight at a given place by means of the relation $-\sin D=\tan\delta\tan\lambda$, the negative sign indicating that when the latitude is north the declination will be south, and *vice versa.*

II. **Solution by W. D. LAMBERT, Washington, D. C.**

This is a slight generalization of Bernoulli's classic problem of the shortest twilight. The expressions that occur in the analytic solution are long and cumbrous, but present no essential difficulty. An outline of the analytic solution is given in Hutton's *Course of Mathematics*, Vol. II, page 385 (12th ed.). The following synthetic solution may be new to many readers.

Suppose for definiteness the body, S, to be west of the meridian. Let P be the pole of the celestial sphere, Z the zenith when the body is in the upper circle of zenith distance $z=ZS$. After the lapse of a period t, the point of the sphere once occupied by Z has moved to Z', and the body S is now in S' at a zenith distance $z+2a=ZS'$. Let ϕ denote the latitude, $=90^\circ-PZ$, and δ the declination $=90^\circ-PS=90^\circ-PS'$. $ZPZ'=SPS'=t$.

The arc ZZ' is found from the isosceles triangle ZPZ' and is given by $\sin\frac{1}{2}ZZ'=\cos\phi\,\sin\frac{1}{2}t$, so that ZZ' increases with t, and will have a minimum value when t has one. From the triangle $ZZ'S'$,

$$\cos ZZ'=\cos z\cos(z+2a)+\sin z\sin(z+2a)\cos ZS'Z'\ldots(1),$$

z and $z+2a$ are given. The only variable on the right is $ZS'Z'$, and since $\sin z$ and $\sin(z+2a)$ are always positive, by increasing $\cos ZS'Z'$ we increase $\cos ZZ'$ and thereby diminish arc ZZ'. ZZ', and with it t, will be a minimum when $ZS'Z'=0$, and then by (1), $ZZ'=2a$, which implies that Z' falls on $S'Z$. From the triangle ZPS',

$$\sin\delta=\cos(z+2a)\sin\phi+\cos\phi\sin(z+2a)\cos PZS'\ldots(2),$$

but from the triangle ZPZ', $\cos PZS'=\tan a\tan\phi$, so that (2) may be reduced to

$$\sin\delta=\frac{\sin\phi\cos(z+a)}{\cos a}\ldots(3),$$

which is the required declination.

The corresponding time is given by

$$\sin\frac{t}{2}=\frac{\sin a}{\cos\phi}\ldots(4).$$

For the problem of shortest twilight, $Z=90^\circ$, and $2a$ is usually taken as 18°. Equations (2) and (3) then show that the central parts of this country, of latitude say 40°, have their shortest twilights when the sun is $5^\circ\ 51'$ south of the equator, or about March 6, and October 8. These twilights last 1 hour, 34 minutes.

It is easily seen from a figure that the parallactic angles at the beginning and end of the minimum period, namely PSZ and $PS'Z$, are equal, and the azimuths PZS and PZS' are supplementary. Some such symmetry as this regard to the prime vertical might be anticipated, for the rate of descent in altitude of a a body is a maximum when on the prime vertical.

Excellent solutions of this problem were received from G. B. M. Zerr, J. Scheffer, and G. W. Greenwood.

$p=\sqrt{(HO_1.O_1K)}=\sqrt{(AD.AF)}\sin\beta$, $\qquad q=\sqrt{(ZO_2.O_2I)}=\sqrt{(AB.AR)}\sin\beta$,

$$AD-AF=FE=\frac{DF\sin\theta}{\cos\beta}, \qquad AB-AR=RC=\frac{BR\sin\phi}{\cos\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{6(p^3DF\sin\theta-q^3BR\sin\phi)}{8(BR\sin\phi-DF\sin\theta)(q^3-p^3)}.$$

Let $DF=2P$, $BR=2Q$. Then

$$p=\frac{P\sqrt{[\cos(\theta+\beta)\cos(\theta-\beta)]}}{\cos\beta}=\frac{P\sqrt{[\cos^2\theta-\sin^2\beta]}}{\cos\beta}, \quad q=\frac{Q\sqrt{[\cos^2\theta-\sin^2\beta]}}{\cos\beta}.$$

$$\therefore \frac{GX}{O_1O_2}=\frac{6[P^4\sin\theta(\cos^2\theta-\sin^2\beta)^{\frac{3}{2}}-Q^4\sin\phi(\cos^2\phi-\sin^2\beta)^{\frac{3}{2}}]}{8\{[Q\sin\phi-P\sin\theta][Q^3(\cos^2\phi-\sin^2\beta)^{\frac{3}{2}}-P^3(\cos^2\theta-\sin^2\beta)^{\frac{3}{2}}]\}}.$$

If the planes of the ellipses are parallel, $\theta=\phi$, and we get

$$\frac{GX}{O_1O_2}=\frac{6(P^4-Q^4)}{8(Q-P)(Q^3-P^3)}=\frac{6(P^2+Q^2)(P+Q)}{P^3-Q^3}.$$

MISCELLANEOUS.

165. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that $\tan^{-1}\frac{n}{n+1}+\tan^{-1}\frac{1}{2n+1}=\frac{1}{4}\pi$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Me.; J. EDWARD SANDERS, Reinersville, O.; FRANK M. DRYZER, A. B., Knoxville, Tenn.; and PROF. J. W. NICHOLSON, State University, La.

Let $\alpha=\tan^{-1}\frac{n}{n+1}$ and $\beta=\tan^{-1}\frac{1}{2n+1}$.

Then $\tan\alpha=\frac{n}{n+1}$, $\tan\beta=\frac{1}{2n+1}$, and $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{\frac{n}{n+1}+\frac{1}{2n+1}}{1-\frac{n}{(n+1)(2n+1)}}=\frac{2n^2+2n+1}{2n^2+2n+1}=1.$$

$\therefore \alpha+\beta=\frac{1}{4}\pi$.

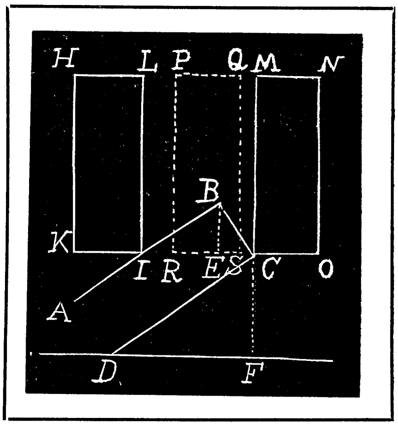
Also solved by G. W. Greenwood.

166. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Several equal rectangular boxes are placed in a row with uniform intervals between the boxes and a passageway along one side of the row. Find the least width of the passageway permitting a box to be removed from the row without moving adjacent boxes. This problem arose during the construction of a room for storage batteries.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $HLIK$, $MNOC$ be two of the boxes, with the box $PQSR$ occupying the space between them in the position $ABCD$. Let $KH=MC=AB=a$, the length of the box; $KI=CO=BC=b$, the width of the box; $IR=SC=c$, the interval between boxes.



When the box is in the position $ABCD$ it can be taken out without removing the adjacent box or boxes.

Then CF is the width of the passageway. $CF=CD\sin CDF=a\sin\theta$.

$$IC=\frac{BC}{\cos\theta}=\frac{b}{\cos\theta}=b+2c.$$

$$\therefore \cos\theta=\frac{b}{b+2c}. \quad \therefore CF=\frac{2a}{b+2c}\sqrt{[c(b+c)]}.$$

$$\therefore \text{Width}=\frac{2a}{b+2c}\sqrt{[c(b+c)]}.$$

If the height of the box is less than the width, and IC wide enough to turn the box on its side, then write b=height and $c=\frac{1}{2}(IC-\text{height})$.

In the above $a>b+2c$, otherwise the passageway would be equal to the width of the box plus a few inches for room.

Also solved by A. H. Holmes.

167. Proposed by DR. OSWALD VEBLEN, Princeton University, Princeton, N. J.

If possible, arrange 43 objects, say the numbers 0, 1, 2, ..., in 43 sets of seven each such that every pair of objects lies in one and only one set of seven. It will then be true that two sets of seven have in common one and only one object.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

The following process of building successive sets of seven each leads to the conclusion that the problem is impossible. Let any set of seven be 0, 42, 41, ..., 37. As no two of this set can occur again together, each of these elements must next be combined with six sets of six elements each, these six elements being chosen from the numbers 36, 35, ..., 1. It will be shown that the twelve sets below may be taken as twelve of the desired sets additional to the above.

$$\begin{array}{l} 42,\ 36,\ 35,\ 34,\ 33,\ 32,\ 31 \\ 42,\ 30,\ 29,\ 28,\ 27,\ 26,\ 25 \\ 42,\ 24,\ 23,\ 22,\ 21,\ 20,\ 19 \\ 42,\ 18,\ 17,\ 16,\ 15,\ 14,\ 13 \\ 42,\ 12,\ 11,\ 10,\ \ 9,\ \ 8,\ \ 7 \\ 42,\ \ 6,\ \ 5,\ \ 4,\ \ 3,\ \ 2,\ \ 1 \end{array} (A), \qquad \begin{array}{l} 41,\ 36,\ 30,\ 24,\ 18,\ 12,\ 6 \\ 41,\ 35,\ 29,\ 23,\ 17,\ 11,\ 5 \\ 41,\ 34,\ 28,\ 22,\ 16,\ 10,\ 4 \\ 41,\ 33,\ 27,\ 21,\ 15,\ \ 9,\ 3 \\ 41,\ 32,\ 26,\ 20,\ 14,\ \ 8,\ 2 \\ 41,\ 31,\ 25,\ 19,\ 13,\ \ 7,\ 1 \end{array} (B).$$

In (A), 42 is any one of the first set, and the remainder of the rows must be all different both within themselves and from each other, as written.

Proceeding to (B), after choosing say, 41, from the first set, each row must contain one and only one element from each row of (A). Since the elements in the several rows of (A) are permutable, it is allowable to take the columns in (A) for the remainders of the rows of (B). For convenience call the result of omitting 42 from (A) the *square.* Further discussion is confined to this square. As the square rotated about its principal diagonal appears in (B), any *new* six must be chosen from it as one would obtain a term from a determinant. But (A) and (B) admit interchanges of both columns and rows, without losing even this reciprocal relation, so that any new six may become the principal diagonal. Hence 36, 29, 22, 15, 8, 1, may be taken as any new six, and is to be coupled with one of the elements 0, 40, 39, 38, 37, to make the fourteenth set of seven. So far 36 has been used three times, hence it must appear in four more sets of six, and the same is true of 29, 22, 15, 8, 1, while none of these six elements may appear in the same set. Thus twenty-four sets of six are accounted for in addition to the above fourteen sets of seven. The remaining sets of six, five in number, will come from the square, with the diagonal omitted.

The square possesses two transformations which are of use in reducing the labor of computing sets of six. The first is a translation along the diagonal, with provision for the edges, and is

$$\begin{matrix} (1,\ 8,\ 15,\ 22,\ 29,\ 36) & (2,\ 9,\ 16,\ 23,\ 30,\ 31) & (3,\ 10,\ 17,\ 24,\ 25,\ 32) \\ (4,\ 11,\ 18,\ 19,\ 26,\ 33) & (5,\ 12,\ 13,\ 20,\ 27,\ 34) & (6,\ 7,\ 14,\ 21,\ 28,\ 35) \end{matrix} \quad (C).$$

The second is the rotation about the diagonal. By a method to be shown, it is possible to find all the consistent sets of four sixes which contain 36. Now (C) leaves invariant the fourteen sets of seven already found, but gives by successive applications to the sets containing 36, those containing 29, then 22, etc. There are fifty-six sets of four sixes for 36, but six of them are invariant under the rotation, coupled with (42, 41), the fourteen sets of seven being also invariant. The remaining fifty sets are, by the rotation, reducible to twenty-five, so that thirty-one sets of four sixes for 36 are to be used. Of course there must be used with these, the entire fifty-six sets for 29, 22, etc.

In computing the sets for 36 the outline is this:

$$\begin{matrix} 36,\ 28,\ (19,\ 20,\ 21,\ 23) \\ 36,\ 27,\ (19,\ 20,\ 23) \\ 36,\ 26,\ (19,\ 21,\ 23) \\ 36,\ 25,\ (20,\ 21,\ 23) \end{matrix} \quad (D), \qquad \begin{matrix} 36,\ 28,\ 19,\ (14,\ 17) \\ 36,\ 27,\ 20,\ (13,\ 16,\ 17) \\ 36,\ 26,\ 21,\ (13,\ 16,\ 17) \\ 36,\ 25,\ 23,\ (14,\ 16) \end{matrix} \quad (E),$$

in which the parentheses in (D) indicate those elements of the third line of the square which are available for the third element of the respective rows in (D). This gives eleven partial results from (D), of which the first is

(E). In this way the total of fifty-six sets of four sixes for 36 is obtained.

The remainder of the work begins with the comparison of such of the thirty-one sets for 36 with each of the fifty-six sets for 29. Although in upwards of forty cases one obtains consistent sets of eight sixes, all of these cases fail at the trial with the sets for 22, showing that the problem is impossible.

PROBLEMS FOR SOLUTION.

ALGEBRA.

283. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Solve $w+x+y+z=4a$, $w^2+x^2+y^2+z^2=4a^2+4b^2$, $w^3+x^3+y^3+z^3=4a^3+12ab^2$, $w^4+x^4+y^4+z^4=4a^4+4b^4+4c^4+24a^2b^2$.

GEOMETRY.

316. Proposed by J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.

Determine the locus of the vertices of parabolas described by particles thrown off from the circumference of a uniformly revolving wheel.

CALCULUS.

239. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

Of all triangles inscribed in a circle, find that which has the greatest perimeter.

MECHANICS.

202. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three equal, uniform, similar rods AB, BC, CD, freely jointed at B and C, are hung from a point by two equal strings attached at A and D. Find the position of equilibrium.

MISCELLANEOUS.

171. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $\lim_{x=a} \dfrac{\phi(x)}{\psi(x)}=\lambda$, show $\lim_{x=a} \left[\dfrac{\lambda}{\phi(x)}-\dfrac{1}{\psi(x)}\right]=\dfrac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}$.

ERRATA.

Page 97, line 10. Vol. XIII, for $x=y=w=$ etc., read $x=x_1=x_2=$ etc.
Page 98, line 1, for $G+D+U$ read $G+D+U+B$, B taken from table.
Page 97, in table add .008 to each number from 34 to 43 inclusive.

THE AMERICAN MATHEMATICAL MONTHLY.

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THE COMPLETE PAPPUS HEXAGON.*

By DR. C. C. GROVE, Hamilton College, Clinton, N. Y.

1. Propositions 138 and 139, Book VII of Pappus Alexandrinus,† as translated by F. Hultsch reads thus:

"Iam his demonstratis ostendendum erit, si parallelae sint $\alpha\beta$ $\gamma\delta$, et in eas incidant quaedam rectae $\alpha\delta$ $\alpha\varsigma$ $\beta\gamma$ $\beta\varsigma$, quarum $\alpha\delta$ $\beta\gamma$ concurrant in μ, et a quovis rectae $\alpha\beta$ puncto inter α et β sumpto ducantur $\epsilon\gamma$ $\epsilon\delta$, quarum $\epsilon\gamma$ cum $\alpha\varsigma$ concurrant in η et $\epsilon\delta$ cum $\beta\varsigma$ in κ, rectam esse quae per $\eta\mu\kappa$ transit."

"At ne sint parallelae $\alpha\beta$ $\gamma\delta$, sed convergant in puncto ν; dico rursus rectam esse quae per $\eta\mu\kappa$ transit."

These theorems Pappus proved by proportion, the equal ratios being respectively between two lines and between the rectangles of two pairs of lines.

Salmon‡ has the same theorems stated thus:

"If ABC are three points of one line and $A'B'C'$ are three points of another line, then the intersections $BC'/B'C$, $CA'/C'A$, $AB'/A'B$ lie on a line."

Different other writers have Pappus's theorem in some wording. The most important mention of the simple case is by Rudolph Boeger,§ who gives it as a simple form, free from the idea of projective relations, of "*Das Sechseck in der Geometrie der Lage.*"

Some other papers directly or indirectly presenting perspective triangles are simply noted:

H. Schroeter: Math. Annalen 2:553—562.

J. Vályi: Archiv der Math. und Physik, 1882, Bd. 70, ss. 105—110; 1884. 2. R. II. T., ss. 230—234.

Rosanes: Ueber Dreiecke in persp. Lage. Math. Ann. 2:549.

Hess: Beitraege z. Theorie d. mehrfach persp. Dreiecke. Ibid. 28: 167.

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*Read before the American Mathematical Society, Chicago Section, April 30, 1907.

†Pappi Mathematicae Collectiones a Federico Commandino Urbinate, in Latin, in U. S Cong. Library. Also, by F. Hultsch in three volumes (Greek and Latin), Berolini, 1877.

‡Conic Sections:—6th Ed., §268, p. 246, Ex. 1.

§*Sechseck und Involution*, Hamburg Mitteilungen, Bd. III., 9, Feb. 1899, s. 387.

Third: Triangles triply in persp. Proc. Edinburg Math. Soc. XIX, p. 10.

L. Klug: Desmische Vierseiten-Systeme. Monatshefte (1903) XIV, s. 74.

M. Pasch: Ueber Vier-eck und seit. Math. Ann. 26:211—216.

Caporali: Memorie, pp. 236, 252.

Veronese: Sull' Hexagrammum mysticum. Lincei Mem. II, 1 (1877), p. 649.

2. The complete figure is constructed thus:

Take six numbers 1—6 and regard them as the names of points or of lines, such that 1, 3, 5 and 2, 4, 6 are three and three respectively on a line or on a point.

We consider the cross-joins as follows:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \\ 5 & 2 & 1 & 4 & 3 & 6 \end{array} \qquad \begin{array}{cccc} (1) & (2) & (3) & [1] \\ (4) & (5) & (6) & [3] \\ (7) & (8) & (9) & [5] \end{array} \left.\right\} \text{on } \Sigma_1 \text{ or } D_1, \text{ respectively.} \left.\right\} \quad (1),$$

$$\begin{array}{cccccc} 3 & 2 & 1 & 4 & 5 & 6 \\ 5 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 5 & 4 & 3 & 6 \end{array} \qquad \begin{array}{cccc} (10) & (11) & (12) & [2] \\ (13) & (14) & (15) & [4] \\ (16) & (17) & (18) & [6] \end{array} \left.\right\} \text{on } \Sigma_2 \text{ or } D_2, \text{ respectively.} \left.\right\} \quad (2).$$

When the six numbers name points, they are given as the points π_i. In that case, [1], etc., are lines to be known as P_i, where $i=1, 2, \dots, 6$. In the dual case, the given six are lines P_i and [1] are points. The scheme means, when points are given, that the line of $\overline{1,2}$ intersects the line of $\overline{4,5}$ in a point (1), that $\overline{2,3}/\overline{5,6}$ is the point (2), and that $\overline{3,4}/\overline{6,1}$ is (3); further, that the points (1), (2), (3) lie on a line [1], and that three lines similarly gotten lie on the point Σ_1. Dualistically, we start with lines $P_{1,3,5}$ on Σ_1 and $P_{2,4,6}$ on Σ_2 and close with points π_i, three on each line D_1 and D_2.

Following the names given to points and lines in the Pascal Hexagon, the points Σ are called Steiner points; the lines P_1, Pappus lines; and the lines D we call Hessian diagonals.

3. From the complete Pappus Hexagon we have the following theorems:

I.

Three lines P_i on each of two points Σ_1, Σ_2, joined by the line S, intersect crosswise in nine points α_{1-3}, β_{1-3}, γ_{1-3}, which join by 18 lines, which are the sides of two sets of three point-triads each.

Three points π_i on each of two lines D_1, D_2, meeting in the point δ, are cross-joined by nine lines which meet in eighteen points, 1—18, which are the vertices of two sets of three line-triads each.

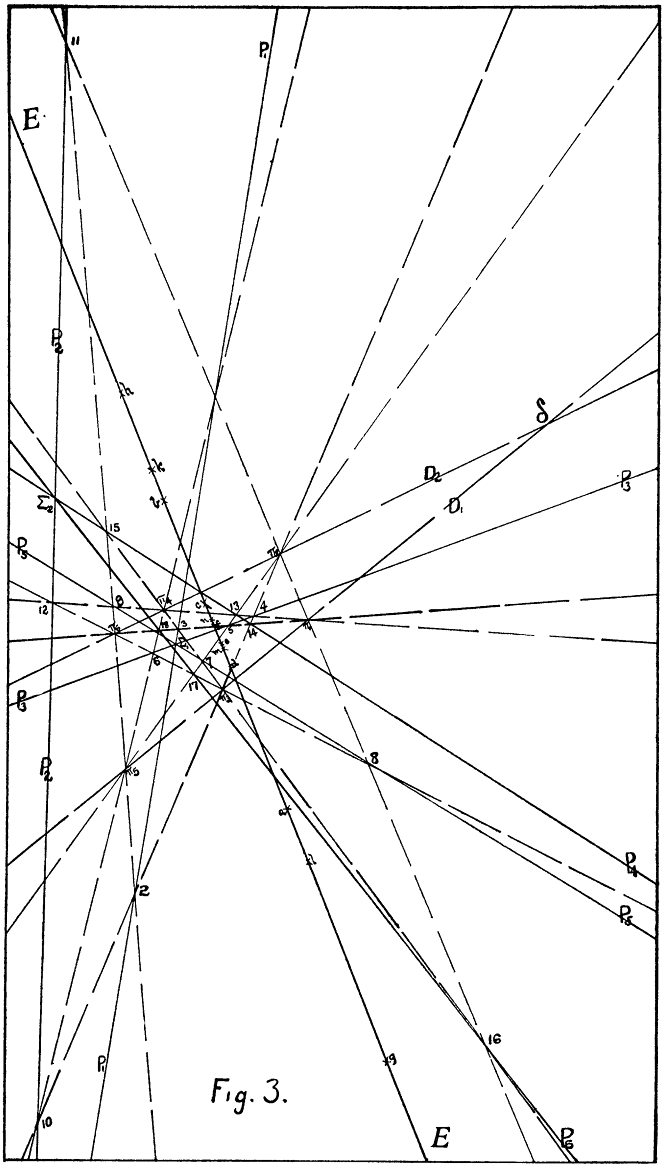


Fig. 1.

The triangles of each set are independently by twos in triple perspective, having as centers of perspective the points Σ_1 and Σ_2 each three lines and three points π_i on the line D_2 or D_1, respectively; and

The triangles of each set are independently by twos in triple perspective, having as axes of perspective the lines D_1 and D_2 each three times and three lines P_i on the point Σ_2 or Σ_1, respectively; and having

having as axes of perspective the line D_1 or D_2 three times each for the sets, respectively, and twelve other lines all of which pass through a point ϵ, which is the pole of the line S as to any of the six triangles. The 18 lines above lie by three on six points π_i, which are three and three on the lines D above.

as axes of perspective the point Σ_1 or Σ_2 three times each for the sets, respectively, and twelve other points all of which lie on a line E, which is the polar of the point δ as to any of the six triangles. The 18 points above lie by three on six lines P_i, which are three and three on the points Σ above.

The frame-work of this theorem as to the triangles and their being in triple perspective is not new but seems a necessary preface to the new parts as to the covariant point ϵ, and line E, and the complete duality that makes the figures really one whole since starting with the three points π_i on each line D_1 and D_2 of the left-hand theorem and following out the right-hand theorem we come back to the original points Σ_1 and Σ_2.

II.

The lines D_1, D_2, and S are the false sides of the complete quadrilateral of the Hessian pairs of the line-triads P_i on Σ_1 and Σ_2.

The points Σ_1, Σ_2, and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point-triads π_i on D_1 and D_2.

From these theorems as also from the demonstration of them, follows the general theorem:

III.

Three lines on each of two points give rise to three points on each of two lines, and the latter by reciprocating the process give rise to three lines on each of the original two points. The derived three lines have the same Hessian pair, or are inclined to each other at the same angles as the original three.

4. We now give a proof of the theorems on the left. Take the two Steiner points, Σ_1, Σ_2, to have co-ordinates σ_1, σ_2, σ_3 and s_1, s_2, s_3, respectively; the triangle $\alpha_1\alpha_2\alpha_3$, formed by the intersections of the Pappus lines as will be shown, as reference triangle; and the line S as auxiliary line.

The line S determined by Σ_1 and Σ_2 is given by

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = 0, \text{ and will be written}$$

$$S_1x_1+S_2x_2+S_3x_3=0.$$

Since this line is taken as auxiliary line, we have the following relations:

$$\left.\begin{array}{c} S_1=S_2=S_3=1, \quad \sigma_1+\sigma_2+\sigma_3=0, \quad s_1+s_2+s_3=0, \\ s_i^2\sigma_j\,\sigma_k-s_j\,s_k\,\sigma_i^2=\sigma_k\,s_k-\sigma_j\,s_j, \end{array}\right\} \qquad (3)$$

where i, j, k are each 1, 2, 3, successively.

The equations of the Pappus lines are the corresponding minors as represented thus:

$$\begin{array}{ccc} P_1 & P_3 & P_5 \\ \hline \sigma_1 & \sigma_2 & \sigma_3 \\ x_1 & x_2 & x_3 \end{array} \qquad \begin{array}{ccc} P_2 & P_4 & P_6 \\ \hline s_1 & s_2 & s_3 \\ x_1 & x_2 & x_3 \end{array} \qquad (4).$$

The nine intersections of the six lines P other than Σ_1 and Σ_2 are named thus:

$$\begin{array}{lccclccc} \text{Line} & P_1 & P_5 & P_3 & & & & \\ \cline{2-4} \text{meets line} & P_2 & P_4 & P_6, & \text{respectively, in point} & \alpha_2 & \alpha_3 & \alpha_1 \\ & P_4 & P_6 & P_2, & & \beta_1 & \beta_2 & \beta_3 \\ & P_6 & P_2 & P_4, & & \gamma_3 & \gamma_1 & \gamma_2 \end{array}\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} \qquad (5).$$

From these co-ordinates by carrying out scheme (1) we find the co-ordinates of the points π_1, π_3, π_5, and of the line D_1 on which they lie. Likewise, following (2) and remembering the relations (3), we find the co-ordinates of the points π_2, π_4, π_6 and of their line D_2. From the co-ordinates of the lines D_1 and D_2 we write the co-ordinates of δ their intersection.

The 9 lines which by threes pass through the points π_i on D_1 are the sides of three point-triads

$$\left\{\begin{array}{cccc} (1) & \alpha_1 & \alpha_2 & \alpha_3, \\ (2) & \beta_1 & \beta_2 & \beta_3, \\ (3) & \gamma_1 & \gamma_2 & \gamma_3; \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{cccc} (1') & \alpha_1 & \beta_1 & \gamma_1, \\ (2') & \alpha_2 & \beta_2 & \gamma_2, \\ (3') & \alpha_3 & \beta_3 & \gamma_3, \end{array}\right\}$$

are the second set of three triangles whose sides pass by threes through the points π_{2i} on D_2.

The triangles of each set are found to be in triple perspective as indicated thus:

Triangles.	*Centers.*	*Axes of Perspective.*
$\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{array}$	π_2, Σ_2, Σ_1.	D_1, $(\sigma_2, \sigma_3, \sigma_1)$, (s_3, s_1, s_2).
$\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array}$	π_4, Σ_1, Σ_2.	D_1, (s_2, s_3, s_1), $(\sigma_3, \sigma_1, \sigma_2)$.
$\begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array}$	π_6, Σ_2, Σ_1.	D_1, $(\sigma_1, \sigma_2, \sigma_3)$, (s_1, s_2, s_3).

$\left.\begin{matrix}\alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2\end{matrix}\right\}$	π_1, Σ_2, Σ_1.	D_2, $(\sigma_2, \sigma_1, \sigma_3)$, (s_2, s_1, s_3).
$\left.\begin{matrix}\alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3\end{matrix}\right\}$	π_3, Σ_2, Σ_1.	D_2, (s_3, s_2, s_1), $(\sigma_3, \sigma_2, \sigma_1)$.
$\left.\begin{matrix}\alpha_3 & \beta_3 & \gamma_3 \\ \alpha_1 & \beta_1 & \gamma_1\end{matrix}\right\}$	π_5, Σ_1, Σ_2.	D_2, $(\sigma_1, \sigma_3, \sigma_2)$, (s_1, s_3, s_2).

The twelve axes other than the D's, evidently pass through a point with co-ordinates (1, 1, 1), which is thus the auxiliary point ε, the pole S as to the reference triangle. Since,

S has the same pole, ε, as to all point-cubics consisting of three joins of the six lines P_i, two and two,	δ has the same polar, E, as to all line-cubics consisting of three joins of the six points π_i, two and two,*
ε is the pole of S as to every and any of the six triangles on all six lines P_i.	E is the polar of δ as to every and any of the six triangles on all six points π_i.

The Lines D_1 and D_2 as Diagonals or False Sides of the Complete Quadrilateral of the Hessian Pairs.

5. The following linear relation exists between the three lines P_i on Σ_1:

$$\sigma_1(\sigma_2 x_3-\sigma_3 x_2)+\sigma_2(\sigma_3 x_1-\sigma_1 x_3)+\sigma_3(\sigma_1 x_2-\sigma_2 x_1)=0.$$

The Hessian covariant of the binary cubic is given by the sum of the squares of these three terms separately, and the imaginary Hessian lines are these terms with the respective coefficients 1, ω, ω^2, and 1, ω^2, ω, in the second case. These two lines and the analogous two on Σ_2 reduce to

$$\sigma_2\sigma_3 x_1+\omega\sigma_3\sigma_1 x_2+\omega^2\sigma_1\sigma_2 x_3=0, \qquad \{1\}$$

$$\sigma_2\sigma_3 x_1+\omega^2\sigma_3\sigma_1 x_2+\omega\sigma_1\sigma_2 x_3=0, \qquad \{2\}$$

The same as {1} with s for σ, {3}

The same as {2} with s for σ, {4}.

These intersect as follows:

{1} and {3} in imaginary point I: $(\sigma_1 s_1, \omega^2\sigma_2 s_2, \omega\sigma_3 s_3)$.

{2} and {4} in imaginary point J: (same with ω and ω^2 interchanged).

{1} and {4} in imaginary point H: $[(\sigma_1 s_1(\sigma_2 s_3-\omega\sigma_3 s_2),\ \sigma_2 s_2(\sigma_3 s_1-\omega\sigma_1 s_3),\ \sigma_3 s_3(\sigma_1 s_2-\omega\sigma_2 s_1)]$.

{2} and {3} in imaginary point K: [same with ω^2 for ω].

*Salmon: *Higher Plane Curves*, §166, pp. 143, 144.

Whence we find that the line of $\overline{IJ}$ is D_1, and that of $\overline{HK}$ is D_2; therefore, D_1 and D_2 are the diagonals of the imaginary quadrilateral of the Hessian pairs of the line-triads on Σ_1 and Σ_2.

6. As may be seen from Fig. 1, for the reverse process we have the triply perspective triangles, thus:

Triangles.	*Centers.*	*Axes of Perspective.*
$\left.\begin{matrix}1 & 6 & 8\\ 2 & 4 & 9\end{matrix}\right\}$	Σ_1, a, b.	P_2, D_2, D_1.
$\left.\begin{matrix}1 & 6 & 8\\ 7 & 3 & 5\end{matrix}\right\}$	c, Σ_1, d.	D_2, P_6, D_1.
$\left.\begin{matrix}2 & 4 & 9\\ 5 & 7 & 3\end{matrix}\right\}$	e, f, Σ_1.	D_2, D_1, P_4.
$\left.\begin{matrix}10 & 14 & 18\\ 16 & 11 & 15\end{matrix}\right\}$	g, Σ_2, h.	D_2, P_1, D_1.
$\left.\begin{matrix}11 & 15 & 16\\ 17 & 19 & 13\end{matrix}\right\}$	k, Σ_2, l.	D_2, P_5, D_1.
$\left.\begin{matrix}10 & 14 & 18\\ 12 & 13 & 17\end{matrix}\right\}$	Σ_2, m, n.	P_3, D_2, D_1.

7. The special forms or arrangements for the three lines on each of two points, to which attention is called, are:

(1) Two sets of equispaced triads, *i. e.*, lines at 120°.

(a) The points being the two equiangular points of the triangles of one set of three.

(b) The points on the circumcircle of equiangular triads.

(2) The points taken at infinity,

(a) Arbitrarily, giving two sets of three parallel lines each, at an angle D_1 with each other.

(b) At I and J, the circular imaginary points, giving two sets of perpendicular lines.

The several forms are handled best by using different co-ordinate systems most convenient for the particular case.

A special form of three points on each of two lines is had when three of the nine lines cross-joining the points by pairs meet in a point either within or without the acute angle at δ. Evidently one triangle of one or of the other set of three becomes a point, which point is then also Σ_1 or Σ_2 and the first six or the second six of the twelve points on E, which thus passes through the same point. All this follows readily in the analysis by observing the relation existing between the co-ordinates when three of the vertices of one triangle are equated.

8. Without further proof because they follow from data already given and may be tested in Fig. 1, we present two more theorems and two others relative to the special forms above.

IV.

A triangle of one set of three is in two-fold perspective with any one of the opposite set of three triangles, but for all such perspectivities—

there are only nine axes each taken twice and situated on the covariant point ε. The centers in each case are Σ_2 and Σ_1.

there are only nine centers each taken twice and situated on the covariant line E. The axes in each case are D_2 and D_1.

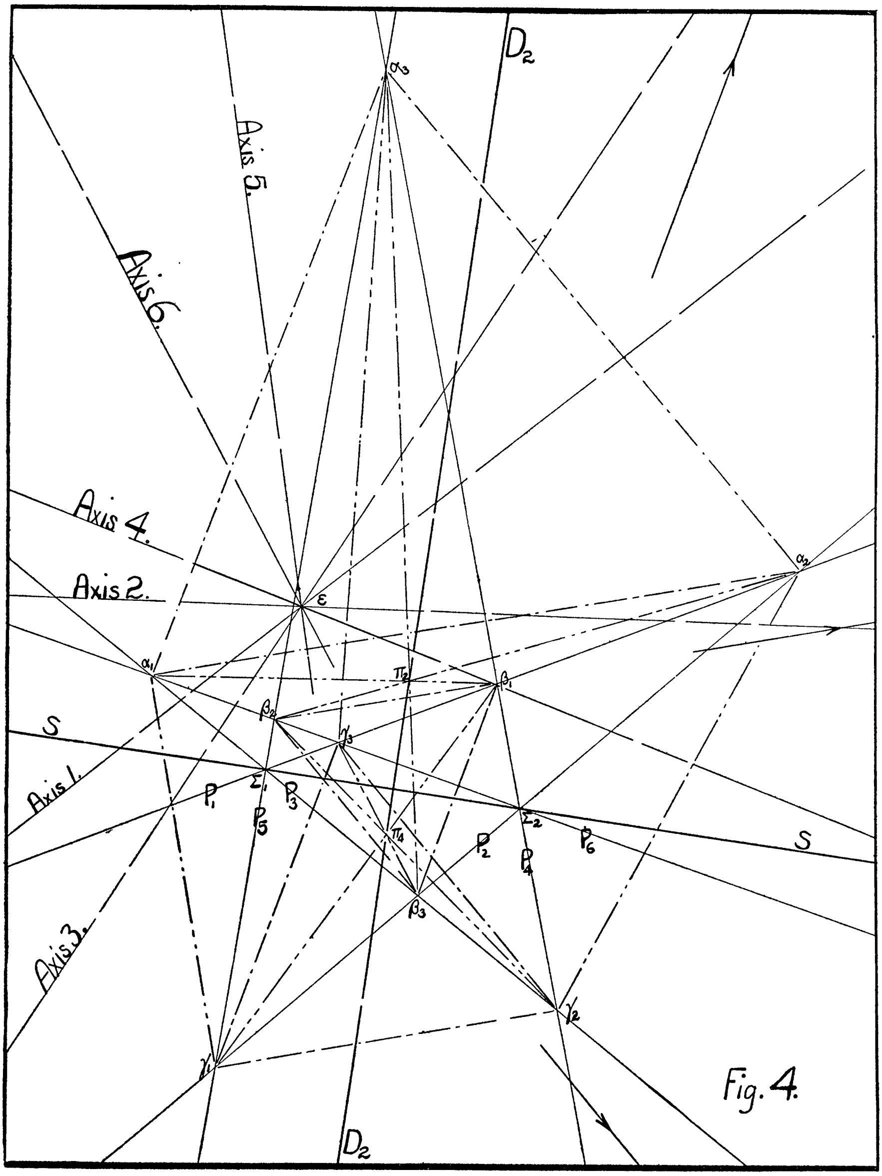


Fig. 2.

V.

The point- and line-triads are between themselves in *single* perspective. The center of perspective in each case is Σ_1 if the two triads are both of the first or both of the second set of three as herein classified, and Σ_2 is center if they are of oppositely named sets.

VI.

For a triad of equispaced lines on each of two points, one set of the three point-triads consists of equiangular triangles with sides respectively parallel (Fig. 2). Thus one of the lines D is at infinity and the other is perpendicular bisector of the line S between Σ_1 and Σ_2. Furthur, the circumcircles of the three equilateral triads pass through Σ_1 and Σ_2, and those of the other set of three triangles intersect in ϵ, the pole of S as to any of the six triangles.

VII.

If to the conditions of the previous theorem we add that one of the set of scalene triangles is also equiangular, then the vertices of the other two triangles of its set are inverse points as to its circumcircle, and all the circumcenters of the set of three equilateral triangles are on the finite Hessian diagonal D.

The proof of this last theorem is especially neat by use of circular co-ordinates.

USING CONJUGATE CO-ORDINATES.

9. As a convenient projection of the hexagon, we take the points π_2, π_4, π_6 on a line, considered the axis of reals, and the points π_1, π_3, π_5 on the line at infinity, so that the lines from the three points on the axis to these three are equispaced lines, parallel, respectively, to

$$x=ty \qquad x=\omega ty \qquad x=\omega^2 ty$$

going, respectively, to $\pi_1 \qquad \pi_3 \qquad \pi_5$

$$\begin{array}{llll} \text{Lines on } \pi_2 \equiv a: x=ty-a(t-1). & x=\omega ty-a(\omega t-1). & x=\omega^2 ty-a(\omega^2 t-1). & \\ \text{Lines on } \pi_4 \equiv b: x=ty-b(t-1). & x=\omega ty-b(\omega t-1). & x=\omega^2 ty-b(\omega^2 t-1). & (6) \\ \text{Lines on } \pi_6 \equiv c: x=ty-c(t-1). & x=\omega ty-c(\omega t-1). & x=\omega^2 ty-c(\omega^2 t-1). & \end{array}$$

The points (p), for $p=1, 2, \ldots, 9$, are in general according to the scheme above,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x=\frac{a-b\,\omega-(a-b)\,\omega^{i+j}\,t}{1-\omega};$$

and the points (q), where $q=10, 11, \ldots, 18$, are,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x=\frac{a-b\,\omega^2-(a-b)\,\omega^{i+j-1}t}{1-\omega^2},$$

or are got from points (p) by interchanging b and c in the equation where $p=q-9$. The same interchange holds throughout the paragraph.

In the above a and b each permute for a, b, c, but a is never b; and ij are each 1, 3, 5, but $i \neq j$ in any one equation. As indicated above a, b, c are the circular co-ordinates of π_2, π_4, π_6 along D_2, the axis of reals.

These 18 points in 6 sets of 3 each, as indicated in schemes (1) and (2), lie on six lines P_i. From the equations of points (p) we get the co-ordinates of lines P.

The lines P_i, for $i=1$, 3, 5, we have the co-ordinates

$$P_i : a+b\omega+c\omega^2, \quad -(a+b\omega^2+c\omega)\omega^{-i}t,$$
$$\omega^2(ab+bc\omega+ca\omega^2)-(ab+bc\omega^2+ca\omega)\omega^{-i+1}t.$$

The co-ordinates of lines P_j, for $j=2$, 4, 6, are

$$P_j : a+b\omega^2+c\omega, \quad -(a+b\omega+c\omega^2)\omega^{-j+1}t,$$
$$\omega(ab+bc\omega^2+ca\omega)-(ab+bc\omega+ca\omega^2)\omega^{-j}t.$$

These lines are evidently equispaced on Σ.

The three lines P_i lie on the point Σ_1, which is

$$x=-\omega^2\frac{ab+bc\omega+ca\omega^2}{a+b\omega+c\omega^2};$$

and the lines P_j lie on the point Σ_2,

$$x=-\omega\frac{ab+bc\omega^2+ca\omega}{a+b\omega^2+c\omega}.$$

These two points are evidently conjugate and therefore symmetrical as to the axis of reals. Further, since the axis of reals, D_2, bisects the line between Σ_1 and Σ_2, the two points Σ are harmonic as to the two lines D.

Since the points Σ are independent of t, these points remain the same for any three equispaced lines on a, b, and c, respectively, mutually parallel; or, keeping one triad of points fixed the triad on the other line may move all along that line provided only that the angles between the lines to them remain constant.

Reversing the Process.

10. The lines P_{ij} intersect in nine points, called before α_{1-3}, β_{1-3}, γ_{1-3}, as follows:

$$P_1P_2 : x=\frac{a^2-bc+(a-b)(a-c)\omega^2t}{2a-b-c}.$$

$$P_1P_4 : x=\frac{c^2-ab+(c-a)(c-b)\omega t}{2c-a-b}.$$

$$P_{16} : x=\frac{b^2-ca+(b-c)(b-a)t}{2b-c-a}.$$

The remaining may be written easily by comparing these with the following:

	P_1	P_2	P_3
P_2	a^2, ω^2	b^2, ω	c^2, 1
P_4	c^2, ω	a^2, 1	b^2, ω^2
P_6	b^2, 1	c^2, ω^2	a^2, ω

The lines joining these intersections in pairs as indicated by schemes (1) and (2) meet by threes in points along D_2 and D_1.

The lines (q) to the *new* points along D_1 have slopes, respectively, t^2, ωt^2, $\omega^2 t^2$; so they turn twice the angle from the axis as the original lines and are like them equispaced.

The lines (p) intersect by threes on three new points, π'_1, π'_3, π'_5, along D_2, corresponding, respectively, with a, b, c if we consider the external segment of D_2. They are

$$x=\frac{\begin{vmatrix} a-b & b^2-ca \\ c-a & c^2-ab \end{vmatrix}}{\begin{vmatrix} a-b & 2b-c-a \\ c-a & 2c-a-b \end{vmatrix}}\ ;\ \text{etc.},$$

taking a, b, c in cyclic order.

11. Thus far the origin on the axis has been arbitrary. Now, considering it the centroid of the three given points, we have $a+b+c=0$, whence also

$$a^2-bc=b^2-ca=c^2-ab=\lambda,\ \text{say};\ bc+ca+ab=-\lambda;\ 2a-b-c=3a,\ \text{etc.}$$

The three new points then become, respectively,

$$x=\frac{a\lambda}{-2bc+ca+ab},\quad x=\frac{b\lambda}{bc-2ca+ab},\quad x=\frac{c\lambda}{bc+ca-2ab}.$$

The counter-triad of the three points a, b, c is

$\dfrac{-2bc+ca+ab}{3a}$ from $(xa/bc)=-1$; call it a'.

$\dfrac{bc-2ca+ab}{3b}$ from $(xb/ca)=-1$; call it b'.

$\dfrac{bc+ca-2ab}{3c}$ from $(xc/ab)=-1$, call it c'.

The cubic along the line a, b, c is $x^3-\lambda x-abc=0$, Differentiating this as to x, we have $3x^2-\lambda=0$, the roots of which are the polar pair of infinity, the intersection of the two lines D_1 and D_2. Calling the roots f and f', we have

$$f+f'=0, \text{ and } ff'=-f^2=-f'^2=-\tfrac{1}{3}\lambda. \quad \therefore f^2=f'^2=\tfrac{1}{3}\lambda.$$

By observation we see then that

$$\pi'_1\, a'=\pi'_3\, b'=\pi'_5\, c'=\tfrac{1}{3}\lambda;$$

so the new triad and the counter-triad of the original triad are in an involution whose double points are the polar pair of the intersection of the lines D_1 and D_2 as to the original triad.

From the results in § 9—11, we see that *in this form* by revolving the three lines on Σ, keeping the triad equispaced, we cut out, at each instant, along D_2 a triad of points having the same Hessian pair and being in the same involution whose double-points are the polar pair of the intersection of the Hessian diagonals as to the original triad.

Thus, we generate a pencil of point-triads along each of two lines, and dualistically of line-triads on each of two points.

THE TRISECTION PROBLEM.

By J. S. BROWN, Southwest Texas State Normal School, San Marcos, Texas.

The solution of this problem by means of the quadratrix, conchoid, and the cardioid are well known, and statements of the fact that the problem has been solved by means of the hyperbolic curve are equally well known, but the writer has never seen a solution by the last named method.

Ball, in his History of Mathematics, says that Viviani solved the problem by means of the *equilateral* hyperbola, and that Vieta determined that its solution depends upon the solution of a cubic equation.

I am not aware that the solution by means of the ceroid [so called from its resemblance to a pair of horns] has ever before been given.

I. Solution by Means of the Hyperbolic Curve.

If a series of circles be drawn through two points A and B, and if BP be one third of the arc BPA and H and H' points in the perpendicular bisector of AB, the locus of the point P, as the circle varies in size, is an hyperbola, since $PB=2PH$ constantly.

As the curve is central, its general equation is

$$a^2y^2-b^2x^2=-a^2b^2, \tag{1}$$

Dividing the members of this equation by b^2 puts it in the form

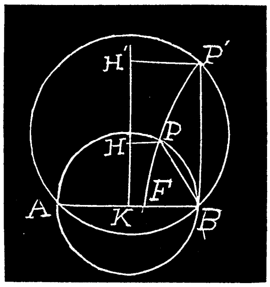
$$\left(\frac{a^2}{b^2}\right)y^2-x^2=-a^2, \tag{2}$$

The factor $\frac{a^2}{b^2}=\cot^2\theta$, θ being the angle which the asymptote makes with the transverse axis. Therefore (2) becomes

$$y^2\cot^2\theta-x^2=-a^2, \tag{3}$$

Since e (eccentricity) $=2$, $\cot^2\theta=\frac{1}{3}$ and θ is therefore 60°. Substituting this value of θ in (3) gives

$$\tfrac{1}{3}y^2-x^2=-a^2, \tag{4}$$



Assuming AB as unity, the circle ABP' as one in which AB is the side of an inscribed square and K the mid point of AB as origin, $P'B=AB=1$, which is the ordinate of the point P', and the abscissa of this point is $a+\frac{1}{3}$.

Substituting these values of y and x in (4), gives

$$\tfrac{1}{3}-(a+\tfrac{1}{3})^2=-a^2, \tag{5}$$

from which $a=\frac{1}{3}$. This value of a substituted in (4) gives $y^2-3x^2=-\frac{1}{3}$, the equation of the above described curve.

To trisect an angle by means of this curve, construct an isosceles triangle upon AB as a base, with the given angle as vertical angle.

The hyperbola will trisect the arc of the circle whose center is the vertex of this angle and whose radius is the leg of the triangle constructed.

II. Solution by Means of the Ceroid.

If a line be drawn through the center O of a circle to meet the circumference and also to meet a straight line LL', let us find the locus of the point P on HO, is HP is constantly equal to KP. Let O be the origin, d the distance from O to LL', $PH=PK=n$, and let r=the radius of the circle. We have

$$\frac{y}{d}=\frac{r+n}{r+2n}, \tag{1}$$

from which is obtained the equation,

$$\frac{y^2}{d^2}(r+2n)^2=x^2+y^2, \tag{2}$$

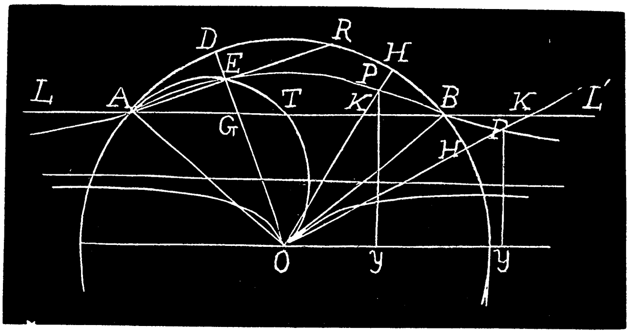
Also, $n=\sqrt{(x^2+y^2)}-r$, and substituting this value of n in (1) gives

$$\frac{y^2}{d^2}[2\sqrt{(x^2+y^2)}-r]^2=x^2+y^2, \tag{3}$$

which is the equation of the required locus.

For convenience of discussion, (3) may be put in form

$$y^2[2\sqrt{(x^2+y^2)}-r-d][2\sqrt{(x^2+y^2)}-r+d]=d^2x^2, \tag{4}$$



It is evident that the curve has two branches which meet at $\pm\infty$ on the asymptote whose equation is $y=\frac{1}{2}d$. n is regarded as negative when *in* the circle, and positive when *outside* of the circle.

The ceroid may be used to trisect any angle. For if AOB is a given angle, and ATO a semi-circle on AO as diameter, and E is the point common to the circumference ATO and the ceroid, the angle AOE is one third of the angle AOB.

Proof. $DE=GE$, for E is on the ceroid.

The angle AEO is a right angle. Hence angle DAE=angle GAE. Then arc AD=arc DR=arc RB. Therefore the angle AOD=one third of the angle AOB.

NOTE ON THE POSTULATE THAT A PART IS EQUIVALENT TO THE WHOLE.

By DR. G. A. MILLER.

The quadratic equations considered in elementary algebra are generally written in one of the following two forms:

$$ax^2+bx+c=0\ldots A,$$
$$ax^2+2bx+c=0\ldots B,$$

where a, b, c are real numbers and a is supposed to differ from zero. An equation of form A is completely determined by the two ratios b/a, c/a. If these two ratios are regarded as the co-ordinates of a point and plotted in the ordinary way with respect to a Cartesian system, every finite point in the plane corresponds to a quadratic equation and every quadratic equation corresponds to such a point. The necessary and sufficient condition that two quadratic equations have different roots is that they correspond to different points. According to this well known interpretation the points of the plane are regarded as representatives of quadratic equations.

The condition that the two roots of an equation of form A are equal to each other is $\frac{b^2}{a^2}=4\frac{a}{c}$. Representing $\frac{b}{a}$ by x and $\frac{a}{c}$ by y this condition becomes $x^2=4y$. Hence the points on the parabola whose equation is $x^2=4y$ represent the quadratic equations of form A whose roots are equal. The points within this parabola, *i. e.*, on the concave side of this curve, represent the quadratic equations whose roots are imaginary, while those on the outside of the parabola represent the equations whose roots are real and distinct. The fact which we desire to emphasize in this connection is that the totality of quadratic equations with real coefficients and imaginary roots can be put into a (1, 1) correspondence with the points within the parabola $x^2=4y$. In other words, the number of the points within this parabola is equivalent to the number of quadratic equations having real coefficients and imaginary roots.

If we suppose that all the equations under consideration are represented by form B, those which have equal roots will be represented by the points of the parabola $x^2=y$. From this it follows that there is a (1, 1) correspondence between the totality of quadratic equations with real coefficients and imaginary roots, and the points within the parabola $x^2=y$. As the number of the points within each of the two parabolas $x^2=4y$ and $x^2=y$ is equivalent to the number of quadratic equations with real coefficients and imaginary roots, the two parabolas must contain the same number of points; *i. e.*, the number of points in one of these parabolas is equivalent to the number of points in the other. As the parabola $x^2=4y$ lies entirely within the one whose equation is $x^2=y$ except where they touch each other, a part of the number of points contained in the parabola $x^2=y$ is equivalent to the total number of these points.

While the above is only one of an indefinite number of illustrative examples of the reasonableness of the postulate that a part of an infinite number is equivalent to the whole, yet it seems to deserve special emphasis in view of its contact with very fundamental matters. Such illustrations seem desirable to prepare the way for the modern definitions of an infinite number; viz., An infinite number is equivalent to a part of itself, or an infinite number remains unchanged if unity is added to it.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Remarks on Problem 179 by TSURUICHI HAYASHI, Koto Shihan Gakko, Tokyo, Japan.

The following question proposed by Dr. L. E. Dickson, The University of Chicago, remains unsolved in Vol. XII, 1905, p. 238:

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

I think that the coefficients of x^2 and x must be interchanged and thus the equation must become

$$x^5+px^2+qx+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

If so, the roots are represented by

$$-\omega_\lambda\sqrt[5]{\frac{q^2}{5p}}+\omega_\lambda{}^3\sqrt[5]{\frac{p^3}{25q}}$$

where $\lambda=1, 2, 3, 4, 5$, and $(\omega_\lambda)^5=1$.

Remark by the PROPOSER.

The problem was printed incorrectly; it should have read $x^5+px^2+qx+\ldots$, with the letters p, q in their natural order. Since p is of the third degree in the roots, and q of the fourth, the constant term is of the fifth degree, as should be the case.

Mr. T. Hayashi's solution of the corrected equation has, doubtless by an oversight in copying, the terms ω and ω^3 interchanged.

Solution. Since the terms x^4 and x^3 are lacking, the simplest expressions to assume for the roots are

$$r_\lambda=\omega_\lambda{}^3A+\omega_\lambda B \quad (\lambda=1, \ldots, 5;\ \omega_\lambda{}^5=1).$$

Then by $\sum\limits_{\lambda=1}^{5}\omega_\lambda{}^r=0$ for $r=1, \ldots, 4$, we have for $S_t=\Sigma r_\lambda{}^t$,

$$S_1=S_2=0,\quad S_3=15AB^2,\quad S_4=20A^3B,\quad S_5=5A^5+5B^5.$$

But for $x^5+px^2+qx+r=0$,

$$S_1=S_2=0,\quad S_3=-3p,\quad S_4=-4q,\quad S_5=-5r.$$

Hence, $A=-\sqrt[5]{\dfrac{q^2}{5p}}$, $B=\sqrt[5]{\dfrac{p^3}{25q}}$, $r=\dfrac{q^2}{5p}-\dfrac{p^3}{25q}$.

276. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $x_1, x_2, \ldots, x_n$ be unequal, and $f(x)$ be a rational integral function of degree $\not> n-2$, then shall

$$\sum_{r=n-1}^{r=1}\frac{f(x_r)}{(x_r-x_1)\,(x_r-x_2)\ldots(x_r-x_n)}=0.$$

Solution by the PROPOSER.

The left hand side written at length is

$$\frac{f(x_1)}{(x_1-x_2)\,(x_1-x_3)\ldots(x_1-x_n)}+\frac{f(x_2)}{(x_2-x_1)\,(x_2-x_3)\ldots(x_2-x_n)}+\ldots$$
$$+\frac{f(x_n)}{(x_n-x_1)\,(x_n-x_2)\ldots(x_n-x_{n-1})}.$$

Let $\dfrac{f(x)}{(x-x_1)\,(x-x_2)\ldots(x-x_n)}\equiv\dfrac{A_1}{x-x_1}+\dfrac{A_2}{x-x_2}+\ldots+\dfrac{A_n}{x-x_n}$.

Then $A_1, A_2, A_3, \ldots, A_n$

$$=\frac{f(x_1)}{(x_1-x_2)\,(x_1-x_3)\ldots(x_1-x_n)},\quad \frac{f(x_2)}{(x_2-x_1)\,(x_2-x_3)\ldots(x_2-x_n)},\ \ldots,$$
$$\frac{f(x_n)}{(x_n-x_1)\,(x_n-x_2)\ \ldots\ (x_n-x_{n-1})}.$$

Hence, $f(x)\equiv\Sigma A_1(x-x_2)\,(x-x_3)\ldots(x-x_n)=$polynomial of degree $\not> n-2$.

Hence, $\Sigma A_1=0$.

This problem, as we thought, proves to be similar to Ex. 4, p. 319, 3rd Edition of Burnside and Panton's *Theory of Equations*. ED. F.

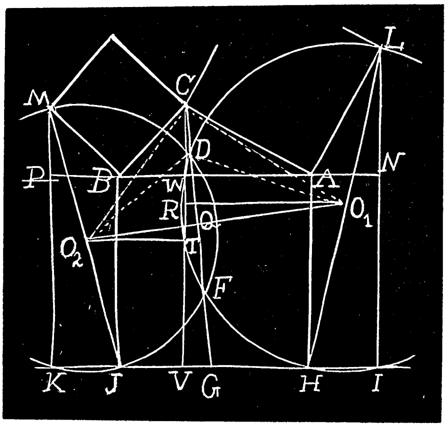
GEOMETRY.

311. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Dallas High School, Dallas, Texas.

Triangle ABC is obtuse-angled at C; x, y, z are squares on the sides AC, CB, BA; LH and MJ are lines joining adjacent sides of x, z and y, z. The common chord of the circles on LH and MJ as diameters passes through C and the mid-point of HJ.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Let ABC be the given triangle with the squares $ABJH$, $BM-C$, $AL-C$ constructed on the sides AB, BC, and AC, respectively. Join M and J and L and H. On LH and MJ as diameters describe the circles whose centers are O_1 and O_2, respectively, the circles intersecting in the points D and F. Draw LI parallel to AH, and meeting JH produced in I. Similarly, draw MK. I and K are on the circumferences of the circles, since angles MKJ and LIH are right angles.



Produce AB to meet MK in P and LI in N. Draw CV perpendicular to JH, and O_1R and O_2T each perpendicular to CV. Draw the common chord DFG and join O_1 and C, O_2 and C, O_1 and D, O_2 and D, D and C, and O_1 and O_2, the line O_1O_2 intersecting DF in Q. Then, from the right triangles O_1QD and O_2QD, we have

$$O_1D^2-O_1Q^2=O_2D^2-O_2Q^2,\text{ or } O_1D^2-O_2D^2=O_1Q^2-O_2Q^2\ldots(1).$$

Now, if FDC is a straight line, we must have

$$O_1C^2-O_1Q^2=O_2C^2-O_2Q^2,\text{ or } O_1C^2-O_2C^2=O_1Q^2-O_2Q^2\ldots(2).$$

Hence, we must have

$$O_1C^2-O_2C^2=O_1D^2-O_2D^2\ldots(3)$$

Now, $O_1C^2-O_2C^2=(O_1R^2+CR^2)-(O_2T^2+CT^2)$

$=(O_1R+O_2T)(O_1R-O_2T)-(CT+CR)(CT-CR)$

$=(\frac{1}{2}AN+AW+WB+\frac{1}{2}BP)(\frac{1}{2}AN+AW-WB-\frac{1}{2}BP)$

$-(CV-RV+CV-VT)(CV-VT-CV+RV)$

$=\frac{3}{2}(AW+WA+CW)(AW-WB)-[2CW-(RV+TV)](RV-TV)$,

since $AW=CW=BP$, $=\frac{3}{2}(AW+WB+CW)(AW-WB)-[2CW+2AB)$

$-\frac{1}{2}(LN+NI+MP+PK)][\frac{1}{2}(LN+NI)-\frac{1}{2}(MP+PK)]$

$=\frac{3}{2}(AW+WB+CW)(AW-WB)-\frac{1}{2}(2CW+\frac{3}{2}AB)(AW-WB)$,

$=\frac{3}{4}(AW^2-WB^2)=\frac{3}{4}(AC^2-BC^2)$.

Also, $O_1D^2-O_2D^2=(O_1D+O_2D)(O_1D-O_2D)=\frac{1}{2}(LN+NI+MP+PK)$

$\times\frac{1}{2}(LN+NI-MP-PK)=\frac{1}{2}(AW+WB)(AW-WB)$,

since $LN=AW$ and $MP=WB$, $=\frac{3}{4}(AW^2-WB^2)=\frac{3}{4}(AC^2-BC^2)$.

Hence, (3) is true and, therefore, GDC is a straight line.

To prove that G is the mid-point of HJ, we have, from the secants GD, GK, and GI, $GK.GJ=GD.GF=GI.GH$, or $GK.GH=GI.GJ$.

Hence, $GK+GH : GH=GI+GJ : GJ$, or $KH : GH=IJ : GJ$.

$\therefore GH=GJ$, since $HI=JK$ and, therefore, $KH=IJ$.

Also solved analytically by G. B. M. Zerr and J. Scheffer. Professors Zerr and Scheffer obtain the equation of the common chord of the two circles and found that its equation is satisfied by the co-ordinates of C and G.

CALCULUS.

236. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the partial differential equation, $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial u}{\partial x}$.

I. Solution by C. EBEN STROMQUIST, Princeton, N. J.

A more general problem, of which the above is a particular case, has been discussed by Darboux (*Leçons sur la Theorie Generale Surfaces*, Vol. 3, p. 53). The solution in this case follows.

Differentiating the given equation with respect to x and setting

$$M=\frac{\partial^2 u}{\partial x^2},\text{ yields } x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}=0\ldots(2).$$

This is a linear differential equation of the first order in M of which a most general solution is* $M=W(x/y)$, where W is an arbitrary function in the argument x/y. Hence

$$w=\int_{x_0}^{x}\int_{x_0}^{x} W\,dx\,dx+x\,w_1(y)+w_2(y)\ldots(3),$$

where w_1 and w_2 are functions of y alone which must be so restricted that w satisfies the given equation.

Since the differential equation from which u is obtained is equation (1) differentiated with respect to x, the result of substituting u in (1) must yield at most a function of y alone. If Y be the function resulting from substituting $u_0=\int_{x_0}^{x}\int_{x_0}^{x} W\,dx\,dx$ in (1), it is readily seen that $Y=0$, and hence that the result of substituting u in (1) gives

$$y\frac{\partial w_1}{\partial y}-w_1=0\ldots(4).$$

Integrating gives $w_1=cy$, whence c is an arbitrary constant.

Substituting in (3), a most general solution of (1) is given by

*See. for instance, Jordan, *Cours d'Analyse*, Vol. 3, p. 314.

$$u=\int_{x_3}^{x}\int_{x_0}^{x} W\left(\frac{x}{y}\right)dx\,dx+cxy+w_2(y),$$

where W and w_2 are arbitrary functions in x/y and y, respectively, and c is an arbitrary constant.

II. **Solution by GEORGE W. HARTWELL, Columbia University, New York City.**

Let $\dfrac{\partial u}{\partial x}=p$. Then the equation will become

$$x\frac{\partial p}{\partial x}+y\frac{\partial p}{\partial y}=p.$$

Solving this by the method of Lagrange we have

$$p=y\,\phi\left(\frac{x}{y}\right)=\frac{y^2}{y}\,\phi\left(\frac{x}{y}\right).$$

Let $\phi\left(\frac{x}{y}\right)$ be the derivative of $\psi\left(\frac{x}{y}\right)$ with respect to $\frac{x}{y}$. Then

$$\frac{1}{y}\,\phi\left(\frac{x}{y}\right)=\frac{\partial}{\partial x}\,\psi\left(\frac{x}{y}\right).\qquad \therefore\frac{\partial u}{\partial x}=y^2\frac{\partial}{\partial x}\,\psi\left(\frac{x}{y}\right).$$

Integrating, $u=y^2\,\psi\left(\frac{x}{y}\right)+\chi(y)$.

III. **Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Multiply the equation by x and let $x=e^v$, $y=e^w$.

Then $\dfrac{d^2u}{dv^2}+\dfrac{d^2u}{dvdw}=2\dfrac{du}{dw}$. Let $\dfrac{d}{dv}=D$, $\dfrac{d}{dw}=D'$. Then $D(D+D'-2)u=0$.

Let $u=e^{av+bw}$. $\therefore\dfrac{du}{dv}=au$, $\dfrac{du}{dw}=bu$.

$\therefore a(a+b-2)=0$, therefore, $a=0$, and $a=2-b$.
$\therefore u=\Sigma Ae^{bw}+e^{2v}\Sigma Be^{b(w-v)}=F(w)+e^{2v}f(w-v)=F(y)+x^2f(y/x)$,
where F and f are arbitrary.

Also solved by the Proposer.

MECHANICS.

197. **Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.**

Suppose that a primary planet and its satellite revolve with uniform angular velocity in circular orbits in the same plane. What relation must hold between the radii of their orbits and their angular velocities in order

that the curve traced by the satellite shall be everywhere concave to the sun? Apply to the earth-moon system to prove that the moon's path is always concave to the sun.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let m=sun's mass, m_1=primary's mass, R=distance of primary from sun, r=distance of satellite from primary, v=velocity of primary around the sun at distance R, v_1=velocity of satellite around the sun at distance R, v_2=velocity of satellite around the primary at distance r.

Then $v^2/R : v_1^2/r = m : m_1$; $v : v_2 = \sqrt{m} : \sqrt{m_1}$; $v_2 : v_1 = \sqrt{r} : \sqrt{R}$.

$\therefore v : v_1 = \sqrt{(mr)} : \sqrt{(m_1R)}$, or $v/R : v_1/r = \sqrt{(mr^3)} : \sqrt{(m_1R^3)}$, the ratio of the angular velocities of primary and satellite in their respective orbits.

Hence, the path of the satellite will be looped, cusped, or direct throughout if

$$\sqrt{(m_1R^3/mr^3)} >=< R/r; \text{ or } m_1R >=< mr, \text{ or } m_1/m >=< r/R.$$

From these, we learn that the path of the satellite will be partly convex, just fail of being convex at perihelion, or be concave, if

$$m_1R^3/mr^3 >=< R/r; \text{ or } m_2R^2 >=< mr^2; \text{ or } m_1/m >=< r^2/R^2;$$
$$\text{or } \sqrt{(m_1/m)} >=< r/R.$$

For the earth and moon, $m/m_1=322,700=(568)^2$, $R=92,000,000$. Hence, if the moon were 92,000,000/322,700=285 miles from the earth, it would travel in a cusped epicycle. If 92,000,000/568=162,000 miles from the earth, the epicycle would be convex. As the actual distance is 238,828 miles, it is always concave.

Also solved by A. H. Holmes.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

141. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given that the highest factor of a prime p contained in $m!$ is p^{m-s}; find general expressions involving p and m and s, from which, when a solution is possible, m can be determined when s is a given integer and p is a given prime. Is it then possible in any case to have more solutions than one?

No solution has been received.

142. Proposed by DR. L. E. DICKSON, The University of Chicago.

Let n be an integer >1 and set $p=n(n-1)+1$. Required n integers whose $n(n-1)$ differences are congruent (modulo p) to the numbers 1, 2, ..., $p-1$. Exhibit at least for n=3, 4, 5, all inequivalent sets of solutions where a set a_1, a_2, ..., a_n is called equivalent to the set $m(a_1-d)$, $m(a_2-d)$, ..., $m(a_n-d)$, for any integers m and d (m not divisible by p).

Solution by DR. C. R. MacINNES, Princeton, N. J.

As Dr. Safford pointed out in discussing No. 132, the problem is equivalent to the following: Find n numbers such that when we add consecutive ones in every possible way, we get all the numbers from 0 to $n(n-1)$. Having any solution of the original problem, without loss of generality we may arrange the numbers in order of magnitude and so get n intervals which add up to $n(n-1)+1$, two of which will be 1 and 2. Such sets can be searched for systematically.

For $n=3$, there is only one set of intervals, 1, 2, 4.

For $n=4$, there are two sets, 1, 2, 6, 4, and 1, 3, 2, 7. But these are equivalent, since multiplying the second set by 2 reproduces the first.

For $n=5$, there is only one set, 1, 5, 2, 10, 3.

For $n=6$, there are five sets, 1, 3, 2, 7, 8, 10; 1, 3, 6, 2, 5, 14; 1, 2, 5, 4, 6, 13; 1, 7, 3, 2, 4, 14; and 1, 2, 7, 4, 12, 5. But these are equivalent, since, if we multiply the solutions corresponding to the first four by 4, 12, 23, and 28, respectively, and add 10, 14, 3, and 3, respectively, we get the last one.

For $n=7$, there is no solution.

For $n=8$, there are six sets of intervals, all of them equivalent to 1, 2, 10, 19, 4, 7, 9, 5.

The original problem, then, has no solution for $n=7$, a unique solution for other values of n up to 8.

n	*Solution.*
3	0, 1, 3.
4	0, 1, 3, 9.
5	0, 1, 6, 8, 18.
6	0, 1, 3, 10, 14, 26.
8	0, 1, 3, 13, 32, 36, 43, 52.

AVERAGE AND PROBABILITY.

179. Proposed by HENRY HEATON, Bellfield, N. D.

Through every point of the circumference of a given circle, chords are drawn in every possible direction. What is their average length?

No solution has been received.

180. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

There are n numbers in a box numbered from 1 to n. A number is drawn and replaced n times. Show that on the average the number of repeats is $\left(\frac{n-1}{n}\right)^n n$.

No solution has been received.

181. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

At a sea-side excursion for x men there are boats enough for q men, and carriages enough for z. But p do not care for driving, and q would feel indifferently comfortable on the water, while the rest do not care either way. Each man has what he prefers as long as a seat is left for him in carriages or boats, and those who do not care either way choose at random. Find the chance that all will be satisfied.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$x<z+q$, $p<q$. Let c=required chance. $x-p-q$ choose at random.

After the $p+q$ persons are satisfied there are still left $q-p$ boats and $z-q$ carriages or $z-p$ conveyances left for the random choosers to select from. $x-p-q$ things can be selected from x things in

$$N=\frac{x!}{(x-p-q)!\,(p+q)!} \text{ ways.}$$

$x-p-q$ things can be selected from $z-p$ things in

$$n=\frac{(z-p)!}{(x-p-q)!\ (z+q-x)!} \text{ ways.}$$

Then $c=\dfrac{n}{N}=\dfrac{(z-p)!\ (p+q)!}{x!\ (z+q-x)!}$.

182. Proposed by L. MORDELL, Philadelphia, Pa.

Out of n straight lines whose lengths are 1, 2, 3, 4, ..., n inches, respectively, the number of ways in which 4 may be chosen which will form a quadrilateral in which a circle may be inscribed is $\frac{1}{48}[2n(n-2)(2n-5)-3+3(-1)^n]$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Taking the first four numbers we get one possible case; taking five, 3; taking six, 7 cases; etc. Thus we have the series 1, 3, 7, 13, 22, 34, 50, 70, 95, 125, ..., of which we have to find the general term. If n is an *even* number, we have the series 1, 7, 22, 50, 95, ..., of which we find $a_0=1$, $\Delta a_0=6$, $\Delta^2 a_0=9$, $\Delta^3 a_0=4$, $\Delta^4 a_0=\Delta^5 a_0 ...=0$. The number of terms is $\frac{1}{2}n-1$.

$\therefore y_n=\frac{1}{24}n(2n^2-9n+10)=\frac{1}{24}n(n-2)(2n-5)$.

If n is an *odd* number, we have the series 3, 13, 34, 70, 125, ..., of which $a_0=3$, $\Delta a_0=10$, $\Delta^2 a_0=11$, $\Delta^3 a_0=4$. Thus, the number of terms being $\dfrac{n-3}{2}$, we find $\frac{1}{24}n(2n^2-9n+10)-\frac{3}{24}=\frac{1}{24}n(n-2)(2n-5)-\frac{3}{24}$.

Both formulae may be condensed into $y_n=\frac{1}{48}[2n(n-2)(2n-5)-3+3(-1)^n]$.

Also solved by G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

284. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k=a_k \\ x^l+y^l=a_l \end{cases} \quad (k,\ l \text{ distinct positive integers})$$

in general and for particular values of $(k,\ l;\ a_k,\ a_l)$.

285. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k+z^k=a_k \\ x^l+y^l+z^l=a_l \\ x^m+y^m+z^m=a_m \end{cases} \quad (k,\ l,\ m \text{ distinct positive integers})$$

in general and for particular values of $(k,\ l,\ m;\ a_k,\ a_l,\ a_m)$.

286. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of n equations in $x_1,\ x_2,\ \ldots,\ x_n$:

$$\begin{array}{l} x_1^{k_1}+x_2^{k_1}+\ldots+x_n^{k_1}=a_1 \\ x_1^{k_2}+x_2^{k_2}+\ldots+x_n^{k_2}=a_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ x_1^{k_n}+x_2^{k_n}+\ldots+x_n^{k_n}=a_n \end{array}$$

where the $k_1,\ \ldots,\ k_n$ are n distinct positive integers, and the $a_1,\ \ldots,\ a_n$ are n given numbers.

GEOMETRY.

317. Proposed by J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.

Find the locus of the vertices of the parabolas described by particles thrown off a uniformly revolving circumference.

318. Proposed by G. W. GREENWOOD, M. A., Roanoke College, Salem, Va.

Is it possible by a straight edge and sect carrier, *i. e.*, without the use of a circle, to construct a mean proportional to two given sects?

CALCULUS.

240. Proposed by L. MORDELL, Philadelphia, Pa.

Show that the osculating conic of the catenary $y=c\cosh\frac{x}{c}$ at the point for which $y=\frac{c\sqrt{10}}{2}$ is a parabola.

241. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Differentiate $y=1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\text{etc.}}}}}$

MECHANICS.

203. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A train weighing $T(=80)$ tons runs first eastward and then westward in latitude $\lambda(=40^\circ)$ at a velocity $v(=45)$ miles an hour. Find the difference between the pressures on the ground in the two cases.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

146. Proposed by PROFESSOR JOSE DE J. CORONADO, Halapa, Vericruz, Mexico.

Find two numbers whose difference is equal to the difference of their cubes.

AVERAGE AND PROBABILITY.

189. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

(a) Lines are drawn from the vertices of a triangle through a random point within it. Find the average area of the triangle formed by joining the points of intersection of these lines with the opposite sides. (b) Lines are drawn from the vertices to points taken at random in the opposite sides of a triangle. Find the average area of the triangle formed by the intersections of these lines.

NOTES AND NEWS.

Mr. S. A. Corey was elected a member of the American Mathematical Society April 27, and was also recently elected a member of the Circolo Matematico di Palermo.

Cardinal Maffi, raised to the purple lately, is a mathematician of considerable note; he started the "Rivista di scienze fisiche e matematiche" and took an active part in the founding of the Italian Catholic Society for Scientific Studies, and in 1903 was made president. J. H. M.

The University of Pennsylvania offers the following advanced courses in mathematics at the summer session, July 8 to August 17, 1907, each course consisting of thirty lectures: Higher Analytic Geometry, by Prof. E. S. Crawley; Definite Integrals, by Prof. I. J. Schwatt; Theory of Functions of a Complex Variable, by Prof. G. H. Hallett; Differential Equations, by Dr. F. H. Safford.

The following courses are to be offered in Collegiate Mathematics at the Summer Session of the University of Illinois: By Dr. Sisam: Modern Geometry, Plane Analytical Geometry. By Dr. Dodd: Functions of Real Variables, Plane Trigonometry. By Mr. Ponzer: Integral Calculus, Teachers' Course in Pedagogy of Secondary Mathematics. By Mr. Emmons: Theory of Equations, College Algebra. All courses are given daily and are equivalent to corresponding courses offered during the regular college year.

The following advanced courses in Mathematics will be offered at the University of Pennsylvania during the year 1907-08: By Professor E. S. Crawley: Solid Analytic Geometry, two hours; Higher Plane Curves, three hours. By Professor G. E. Fisher: Theory of Functions of a Complex Variable, first half year, three hours; Elliptic Functions, second half year, three hours. By Professor I. J. Schwatt: Definite Integrals, three hours. By Professor G. H. Hallett: Lie's Theory of Continuous Groups, first half year, three hours; Galois' Theory of Algebraic Equations, second half year, three hours. By Dr. F. H. Safford: Curvilinear Co-ordinates, three hours. By Dr. O. E. Glenn, Higher Algebraic Equations, two hours.

The following courses in mathematics are to be offered at the University of Wisconsin during the summer session, 1907: By Professor Van Vleck: Solid Geometry, admission credit; Geometry, a general survey of the progress of geometry, two hours' credit; Theory of Point Sets, two hours' credit. By Professor Dowling: Introduction to Higher Plane Curves, one hour's credit; Differential and Integral Calculus, two hours' credit; Plane Trigonometry, two hours' credit. By Professor Skinner: Elementary Algebra, admission credit or two hours' credit. By Assistant Professor Mason: Analytic Geometry, two hours' credit; Application to Mechanics, one hour's credit; Number Concept and Geometric Construction, one hour's credit.

The followlng courses in mathematics are to be offered at the University of Michigan during the summer session, 1907: By Professor Beman: Differential Equations, two hours' credit; Geometry and Algebra, a course for teachers, two hours' credit. By Professor Markley: Theory of Functions of a Complex Variable, two hours' credit; Projective Geometry, two hours' credit; College Algebra, two hours' credit. By Assistant Professor Glover: Elementary Algebra, admission credit; Theory of Annuities and Insurance, two hours' credit. By Dr. Running: Analytic Mechanics, two hours' credit; Trigonometry, two hours' credit. By Dr. Field: Differential Calculus, two hours' credit; Integral Calculus, two hours credit. By Mr. Escott: Plane Geometry, admission credit; Analytic Geometry, four hours' credit.

BOOKS AND PERIODICALS.

Elements of the Infinitesimal Calculus. By G. H. Chandler, M. A., Professor of Applied Mathematics, McGill University, Montreal. Third Edition, rewritten. 12mo Cloth, vi+319 pages, 146 figures. Price, $2.00. New York: John Wiley & Sons.

This work is intended as an introductory manual of the Calculus for beginners generally and students of engineering particularly. The author recognizes the Theory of Limits as the logical foundation of the Calculus, but in his book he aims to accustom the reader to the principles and processes which are used in practical operations. The arrangement of topics is somewhat out of the usual order. Thus the differentials of Hyperbolic Functions are discussed immediately after the differentials of the ordinary functions. Also integration is taken up early, page 93. Some subjects not usually treated in elementary texts find a brief treatment in this work. Thus the Intrinsic Equations of Curves, Fourier Series, and Differential Equations are treated, Fourier Series occupying pages 228-238 and Differential Equations 259-286. Elliptic Integrals are also briefly discussed. Many of the problems are related to practical application and there are many illustrating the various subjects under discussion. At the end of the book there are the following tables: (1) Powers, Neaperian Logarithms, etc. (2) Circular Functions; Hyperbolic Functions; Lambda Functions: Gamma Functions; First Ellliptic Integrals; and Second Elliptic Integrals. B. F. F.

Higher Mathematics for Students of Chemistry and Physics, with Special Reference to Practical Work. By J. W. Mellor, D. Sc. Second Edition, enlarged. 8vo Cloth, xxi+631 pages. Price, $4.50 net. New York: Longmans, Green, & Co.

The author has rendered a good service to the students of chemistry and physics by preparing for their use this excellent work. In the establishment of principles and rules, the author has kept constantly in mind the class of students for which he was preparing his book, and thereby he has avoided the extreme logical rigor and and subtle formalities on the one hand, while at the same time no loose and illogicel reasoning is permitted, on the other.

The student will in the study of this work get a good grounding in the Differential and Integral Calculus, Infinite Series, Differential Equations, Theory of Error, Calculus of Variations, etc. While some of the problems belong to the domain of pure mathematics and are to be found in many text-books, yet the greater number are based upon measurements, etc, recorded in current scientific journals.

The style of type, printing, and binding are excellent, and the work is one that will commend itself favorably to all classes of mathematical students. B. F. F.

The Electron Theory. A Popular Introduction to the New Theory of Electricity and Magnetism. By E. E. Fournier D'Albe, B. Sc. (London), A. R. C. Sc., Compiler of "Contemporary Electrical Science" with a Preface, by G. Johnstone Stoney, M. A., Sc. D., F. R. S. 8vo Cloth. xxiii+311 pages. Price $1.50 net. New York: Longmans, Green, & Co.

In this work is set forth in popular language the latest theory regarding the nature of matter. All students of science unacquainted with the most recent advances in Electricity and Magnetism, will want this book. B. F. F.

Wellcome's Photographic Exposure Record and Diary. Price, 50 cents. London and New York: Burroughs, Wellcome, & Co.

This little book gives a clear explanation of the underlying principles of exposure and puts them into practice by means of a very simple mechanical calculator which indicates the correct exposure. The book is a compact compendium of photographic information. In addition it provides a pocket note-book, a diary and ruled pages for recording exposures. This little book will be of great service to the amateur photographer. B. F. F.

New Elementary Arithmetic. By George Wentworth, Author of Text-Books in Mathematics. 8vo Cloth, 232 pages. Boston and Chicago: Ginn & Co.

This book is designed to be taken up not later than the last half of the second school year. The work, in addition to well selected material for the pupils, contains also some good suggestions for the teacher. B. F. F.

Theoretical Mechanics. By J. H. Jeans, Fellow of Trinity College, Cambridge (England), and Professor of Applied Mathematics in Princeton University. 8vo Cloth, 364 pages. Illustrated. List price, $2.50; mailing price, $2.65. Boston and Chicago: Ginn & Co.

This book is intended to supply a one year's course for students beginning the study of mechanics. It treats of the general principles of dynamics, the laws of motion, statics, and dynamics of a particle of a rigid body. The treatment of the subject is lucid and each principle is illustrated by a series of practical examples. Numerous well selected exercises and problems are inserted for solution by the student. B. F. F.

Plane Geometry. By Edward Rutledge Robbins, A. B., Senior Mathematical Master, The William Penn Charter School. 8vo, Cloth sides, Leather back, 254 pages. New York, Cincinnati, and Chicago: American Book Co.

Among the reasons given by the author in preparing this work are: To present a book that has been written for the pupil; to stimulate his mental activity; to present a text that will be clear, consistent, teachable, and sound; and to explain rather than formally demonstrate the simple fundamental truths. The diagrams are good, the typographical and mechanical execution of the book is splendid, and the book is well written. A large number of original exercises is inserted. B. F. F.

Physics. By Charles Riborg Mann, the University of Chicago, and George Ransom Twiss, the Central High School, Cleveland. 8vo Cloth, 458 pages. Chicago: Scott, Forsman, & Co.

This book is so written and the subject matter presented in such a way that the attention of the pupil is held to the subject under discussion. The illustrations are new, being, for the most part, pictures of real objects. The most recent discoveries in Physics are here clearly described and the underlying principles fully explained. This work will go far towards stimulating a greater interest in the study and teaching of Physics. B. F. F.

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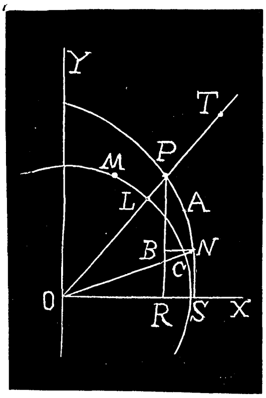
ON DIVIDING AN ANGLE INTO PARTS HAVING THE RATIOS OF ANY GIVEN STRAIGHT LINES.*

By REV. R. D. CARMICHAEL, Presbyterian College, Anniston, Alabama.

The solution of this problem will be here effected by aid of the locus of the polar equation $\rho\sin\theta=m\theta$. In a similar manner it may be carried out by means of the locus of $\rho\cos\theta=m\theta$. We shall take the special case $m=1$ of the first equation, thus giving $\rho\sin\theta=\theta$.

It is desirable to have a method of constructing the curve by continuous motion. We proceed in the following manner: Construct a material circle with center O and radius unity, as in the figure, and let it be fixed to the plane of the paper. Let O be the origin and OX the polar axis. Let P be any point on the curve, and let OPT be a straight bar pivoted to the paper at O and free to move without carrying the circle with it. Draw PR perpendicular to OX; now we have $OP=\rho$, $\angle POR=\theta$, and $PR=\rho\sin\theta$. Then from the equation of the curve, $PR=\theta$. That is, PR is numerically equal in length to the arc LCS. Now fasten one end of a cord at some point Q in the circumference of the circle and pass it any convenient number of times around the circle in the direction QLS. On the last round let the free end of the cord pass through a small ring fastened to OT at L. Then let it pass around a roller at T and back through another small ring at P, with its free end taking the direction PR perpendicular to OX at R, the extremity of the cord being at R. Take now another chord of length $LT+PT$, with its ends attached to the rings at L and P, the cord also passing around the roller at T. As the line OT moves, let the roller at T so slide as to keep the last chord tightly stretched. (This may be accomplished by having T attached to a spring.) The purpose of this chord is to make the distance $LT+PT$ remain canstant as OT moves. If a pencil is also fastened



*Presented to the American Mathematical Society, February 23, 1907.

to the ring at P and passed through a slit in OT, it will furnish a means of marking on the paper the path of P. If OT moves toward OY so that L comes to any point as m, a portion of the first cord equal to the arc LM will be set free. Now if P, R, and T should also move so that they remain in the positions defined above, the pencil at P will trace on the paper an arc of the locus, as may be very easily proven. PR and OX may be kept at right angles by means of a sliding right triangle with one leg in RX and the other in RP. The proper relative positions of L, P, and T are easily found when OT coincides in direction with OX. Hence the method given is a complete solution of the problem of the construction of the curve by continuous motion.

We shall now apply this locus to the problem of dividing an angle into parts having the ratios of any given straight lines. Let the given angle take the position POS in the figure. If the line PR is divided in the given ratio (as it is always easy to do) and through the points of section lines are drawn parallel to OX intersecting the arc PNS in the points A, N, etc.; then if the lines OA, OM, etc., are drawn, they will divide the angle POS into the required parts.

PROOF. Let B be one of the points of section on PR as defined above. Draw BN parallel to OX. Draw ON. It is evidently sufficient to our demonstration to prove that

$$\angle PON : \angle NOS :: PB : BR;$$

for then we can evidently cut off from one side of the angle one of the proportional parts; from a side of this another; and so on. This clearly gives the same points A, N, etc., as above. A perpendicular let fall from N on OX is equal to BR. Hence from the method of constructing the locus, we know that

the line BR is numerically equal to the arc CS.

Also, the line PR is numerically equal to the arc LS.

Hence, $$\text{arc } LS : \text{arc } CS :: PR : BR.$$

Then, by division of ratios,

$$\text{arc } LC : \text{arc } CS :: PB : BR.$$

Hence, $$\angle PON : \angle NOS :: PB : BR.$$

Therefore, the problem is completely solved by means of the locus of $\rho \sin \theta = \theta$.

REMARK. A special case of this is the problem of dividing an angle into n equal parts. It will be noticed that in the present case it is not necessary to construct different curves for different values of n, but that one

curve gives the solution for every case. Attention is also called to the fact that by this method an angle may be divided in any ratio or ratios in which a straight line may be divided. It is therefore as easily separated into parts having an incommensurable ratio as into any other, provided that that incommensurable ratio can be expressed by means of straight lines.

NOTE ON THE VOLUME OF A TETRAHEDRON IN TERMS OF THE COORDINATES OF THE VERTICES.

By DR. L. E. DICKSON.

1. Quite a variety of propositions of solid analytic geometry are needed for the usual derivation of the volume of a tetrahedron (cf. C. Smith, p. 24). If, as in the present note, we give an elementary proof making use merely of the concept of coordinates, we are in a position to apply the result to derive* easily several of the initial propositions in solid analytics, *e. g.*, that the equation of any plane is of the first degree, and conversely.

The plan of the proof (§ 3) is entirely obvious. The only novelty lies in a certain device which yields the result without computation. This device will first be illustrated in deriving the area of a triangle (§ 2).

2. Let the vertices of a triangle $\triangle$ taken in counter-clockwise order be (x_1, y_1), (x_2, y_2), (x_3, y_3). Then $\triangle$ can be expressed in terms of three right trapezoids with parallel sides y_i. The area of a right trapezoid with parallel sides y_1 and y_2, and base b, is $\frac{1}{2}b(y_1+y_2)$, being half of the rectangle of height y_1+y_2 and base b. Hence

$$2\triangle=(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+(x_3-x_1)(y_3+y_1).$$

The device consists in setting $s=y_1+y_2+y_3$. Then

$$2\triangle=(x_1-x_2)(s-y_3)+(x_2-x_3)(s-y_1)+(x_3-x_1)(s-y_2).$$

Since each x occurs once positively and once negatively, the terms in s evidently cancel. The remaining terms give the expansion, according to the second column, of

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

*For plane analytics, this plan is followed in the chapter on graphic algebra in the writer's *College Algebra* (John Wiley and Sons).

3. Consider any tetrahedron $T=P_1P_2P_3P_4$, the notation for the vertices being chosen so that P_1 is above the plane of P_2, P_3, P_4, while the latter lie in counter-clockwise order when viewed from P_1. Denote P_i by (x_i, y_i, z_i); its projection on the x, y-plane is $Q_i=(x_i, y_i, 0)$. Now T can be expressed in terms of four truncated right triangular prisms $P_iP_jP_kQ_iQ_jQ_k$, i, j, k denoting three of the four numbers 1, 2, 3, 4. The area of $Q_iQ_jQ_k$ is given by a determinant (§ 2). Applying the formula (§ 4) for the volume of a truncated right prism, we get

$$6T=D_4(z_1+z_2+z_3)+D_3(z_1+z_2+z_4)+D_2(z_1+z_3+z_4)-D_1(z_2+z_3+z_4),$$

where

$$D_1=\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix},\ D_2=\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix},\ D_3=\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_2 & y_2 & 1 \end{vmatrix},\ D_4=\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The device consists in setting $s=z_1+z_2+z_3+z_4$. Then

$$6T=D_4(s-z_4)+D_3(s-z_3)+D_2(s-z_2)-D_1(s-z_1).$$

Here the terms free of s equal the expansion, according to the third column, of

$$D\equiv\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The terms multiplying s are derived from the others by replacing each z by -1 and hence equal the expansion of a determinant derived from $-D$ by replacing each z by 1. But a determinant with two columns alike vanishes identically. Hence $T=\frac{1}{6}D$.

4. The volume of a truncated right prism P, whose base is a triangle $\triangle$, and lateral edges are a, b, c, is $\frac{1}{3}(a+b+c)\triangle$. This may be proved as in the geometries, or very simply as follows. Let $a\geqq b\geqq c$. Let the edge c be DE, d the side of triangle opposite D, h the perpendicular from D to d, so that $\triangle=\frac{1}{2}hd$. The plane through E parallel to $\triangle$ divides P into a right prism of volume $c\triangle$ and a pyramid with summit E, altitude h, and base a trapezoid with parallel sides $a-c$, $b-c$, and common perpendicular d. The area of the trapezoid is $\frac{1}{2}d(a+b-2c)$; the volume of the pyramid is therefore $\frac{1}{3}\triangle(a+b-2c)$. Adding $c\triangle$ to the latter, we get $P=\frac{1}{3}\triangle(a+b+c)$.

To give another proof, extend b (upwards) the length $a-b$. Thus to P we add a triangular pyramid with summit E, and base a right triangle of lege $a-b$ and d, and hence of volume $\frac{1}{2}d(a-b).\frac{1}{3}h=\frac{1}{3}\triangle(a-b)$. Next, extend c the length $a-c$. We thus add on a pyramid of summit E, and base equal to $\triangle$, and hence of volume $\frac{1}{3}\triangle(a-c)$. By these additions, P becomes a right prism of volume $\triangle a$. Hence

$$P+\tfrac{1}{3}\triangle[a-b]+\tfrac{1}{3}\triangle[a-c]=\triangle a,\quad P=\tfrac{1}{3}\triangle[a+b+c].$$

AN EXAMPLE OF THE INDICATRIX IN THE CALCULUS OF VARIATIONS.

By ARNOLD DRESDEN, The University of Chicago.

§ 1. INTRODUCTION.

Suppose there is given a definite integral

$$I=\int_{t_1}^{t_2} F(x, y, x', y')dt, \tag{1}$$

in which x and y are functions of some parameter t, x' and y' are the derivatives of these functions with respect to t. Let the function F be continuous and have continuous derivatives of the first, second and third order in a domain T of the variables x, y, x', y', defined by (x, y) in a region R of the xy-plane, x' and y' finite and restricted by the condition

$$x'^2+y'^2\neq 0. \tag{2}$$

The definite integral (1) may now be taken along an infinitude of curves between $P_1[x(t_1), y(t_1)]$ and $P_2[x(t_2), y(t_2)]$ in the domain T.

The simplest problem of the Calculus of Variations is to determine among the totality of all these curves, restricted by certain conditions, the one for which I is a *maximum*, or a *minimum**. We shall use the word *extremum* to denote either maximum or minimum.

Let now

$$x=\phi(t), \quad y=\psi(t), \quad t_1 \overline{\gtrless} t \overline{\gtrless} t_2, \tag{3}$$

be the equations in parameter-representation† of a curve which minimizes (1). If we restrict ourselves to functions ϕ and ψ, which have continuous first derivatives, the conditions that (3) furnishes a weak‡ minimum for (1) are§

1. *The functions $\phi(t)$ and $\psi(t)$ must satisfy Euler's differential equation, in the Weierstrass-form, i. e.,*

$$\overline{F}_{x'y}-\overline{F}_{xy'}+\overline{F}_1[\phi''\psi'-\phi'\psi'']=0. \tag{I}$$

The function F_1 is defined by

$$F_1(x, y, x', y')=\frac{F_{x'x'}(x, y, x', y')}{y'^2}=-\frac{F_{x'y'}(x, y, x', y')}{x'y'}=\frac{F_{y'y'}(x, y, x', y')}{x'^2}. \tag{4}$$

*For an exact formulation of the problem see O. Bolza, Lectures on the Calculus of Variations, §3.

†See C. Jordan, Cours d'Analyse, Vol. I, 2nd ed., pp. 90-108.

‡See O. Bolza, loc. cit., pp. 69-71.

§Ibid., Chapter IV.

The stroke over the function-symbols is used to denote that the arguments are to be taken as follows:

$$x=\phi(t), \quad y=\psi(t), \quad x'=\phi'(t), \quad y'=\psi'(t).$$

Any curve, satisfying (I), will be called an *extremal.**

II. *Legendre's condition must be fulfilled, i. e.,*

$$\bar{F}_1 \geqq 0, \quad t_1 \overline{\overline{<}} t \overline{\overline{<}} t_2.$$

III. *Jacobi's condition must be satisfied, i. e.,*

$$t_2 \leqq t_1'.$$

t_1' denotes here the parameter-value of the conjugate-point† of $P(t_1)$.

If the minimum shall be strong‡, a fourth condition needs to be satisfied:

IV. $E(x, y; x' y'; \bar{x}', \bar{y}') \geqq 0$, for every point along (3) and for any pair of finite values of $\bar{x}', \bar{y}'$ different from x', y', and restricted by the condition:

$$\bar{x}'^2+\bar{y}'^2 \neq 0.$$

The function E is defined by:

$$E(x, y; x', y'; \bar{x}', \bar{y}')=\bar{x}'[F_{x'}(x, y, \bar{x}', \bar{y}')-F_{x'}(x, y, x', y')]+$$
$$+\bar{y}'[F_{y'}(x, y, \bar{x}', \bar{y}')-F_{y'}(x, y, x', y')].$$

A stronger form of (IV) is

(IV') $\qquad F_1(x, y, \cos\gamma, \sin\gamma) \geqq 0$ along (3), and for $0 \leqq \gamma \leqq 2\pi$.

A curve, satisfying condition (IV'), shall be called a *hyperstrong minimum.*

The conditions for a weak, strong, or hyperstrong maximum are obtained out of (I), (II), (III), (IV), and (IV') by replacing the inequality signs by the opposite ones.

The conditions, as stated above, with the inclusion of equality signs, are the conditions for *improper extrema.*§

By omitting the equality signs from (II), (III), (IV), and (IV'), we obtain the conditions for *proper extrema.*§

If we free ourselves of the restriction, that the functions $\phi'(t)$ and $\psi'(t)$ should be continuous, and admit curves with corners (so called *discon-*

*See O. Bolza, loc. cit., p. 27, p. 123.
†See O. Bolza, loc. cit., p. 60, p. 135.
‡See O. Bolza, loc. cit., p. 69, p. 71.
§See O. Bolza, loc. cit., p. 11.

*tinuous solutions**) as solutions of our problems, still another condition must be satisfied:

(V) *Weierstrass' corner-condition.*† At every corner we must have:

$$\left.\begin{aligned} F_{x'}(x, y, x', y') &= F_{x'}(x, y, \underline{x}', \underline{y}') \\ F_{y'}(x, y, x', y') &= F_{y'}(x, y, \underline{x}', \underline{y}') \end{aligned}\right\}, \tag{V}$$

where x', y', and $\underline{x}'$, $\underline{y}'$ denote the forward and backward derivatives of $\phi(t)$ and $\psi(t)$ at the corner.

2. Carathéodory‡ has given a method by means of which we can decide whether or not the conditions (II), (IV), (IV'), and (V) are satisfied by a given extremal.

For every point of the region R, in which the function F, and its derivatives of first, second, and third order are determined, he defines a curve, called the *Indicatrix*, by means of its equation in polar coordinates:

$$\rho = \frac{1}{F(x, y, \cos\theta, \sin\theta)}, \tag{5}$$

the origin of coordinates being a point G. We make the usual convention of Analytic Geometry, that, if $\rho<0$, for $\theta=\theta_1$, the absolute value of ρ shall be laid off on the half-line $\theta=\theta_1+\pi$.

He proves then the following theorems:§

A. If $F(x, y, \cos\theta, \sin\theta)$ is of constant sign for $0 \leqq \theta \leqq 2\pi$, the indicatrix is a closed curve around the origin G.

Proof for A. The theorem follows at once from the definition of the Indicatrix and the usual conventions concerning polar coordinates.

B. If $F(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0$, and the indicatrix has positive curvature at $\theta=\theta_0$,

$$F_1(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0.$$

If $F(x, y, \cos\theta_0, \sin\theta_0) \gtrless 0$, and the indicatrix has negative curvature at $\theta=\theta_0$,

$$F_1(x, y, \cos\theta_0, \sin\theta_0) \lessgtr 0.$$

Proof for B. Referring the Indicatrix to rectangular coordinates:

$$\xi=\rho\cos\theta, \quad \eta=\rho\sin\theta,$$

we find

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*See O. Bolza, loc. cit., p. 36.

†Ibid., p. 38, p. 126.

‡C. Caratheodory, *Ueber die discontinuierlichen Loesungen in der Variationsrechnung,* Gottingen, 1904, p. 69; *Mathematische Annalen,* Vol. 62, p. 456.

§*Mathematische Annalen,* Vol. 62, pp. 457, 460, 461, 465.

$$\xi = \frac{\cos\theta}{F(x,\,y,\,\cos\theta,\,\sin\theta)},\qquad \eta = \frac{\sin\theta}{F(x,\,y,\,\cos\theta,\,\sin\theta)}. \tag{6}$$

From this, making use of the formulae:

$$F(x,\,y,\,\cos\theta,\,\sin\theta) = F_{x'}\cos\theta + F_{y'}\sin\theta,$$

$$F'(x,\,y,\,\cos\theta,\,\sin\theta) = -F_{x'}\sin\theta + F_{y'}\cos\theta,^{*}$$

and the definition of the F_1-function, we obtain:

$$\xi' = -\frac{F_{y'}}{F^2},\qquad \eta' = \frac{F_{x'}}{F^2} \tag{7}$$

and

$$\xi'' = \frac{2F'F_{y'} - FF_1\cos\theta}{F^3},\qquad \eta'' = -\frac{2F'F_{x'} + FF_1\sin\theta}{F^3},$$

where $'$ indicates differentiation with respect to θ. Consequently:

$$\frac{d^2\xi}{d\eta^2} = \frac{\xi'\eta'' - \xi''\eta'}{\xi'^2} = \frac{F_1}{\xi'^2 F^3};$$

from which the theorem follows at once.

C. If $F(x,\,y,\,\cos\theta_0,\,\sin\theta_0) \gtrless 0$, and G and $\bar{Q}$ lie on the same side of the tangent to the indicatrix at $Q(\theta_0)$,

$$E(x,\,y;\,\cos\theta_0,\,\sin\theta_0;\,\cos\bar{\theta}_0,\,\sin\bar{\theta}_0) \gtrless 0.$$

If $F(x,\,y,\,\cos\theta_0,\,\sin\theta_0) \gtrless 0$, and G and $\bar{Q}(\bar{\theta}_0)$ lie on opposite sides of the tangent to the indicatrix at $Q(\theta_0)$,

$$E(x,\,y;\,\cos\theta_0,\,\sin\theta_0;\,\cos\bar{\theta}_0,\,\sin\bar{\theta}_0) \lessgtr 0$$

(see Fig. 1).

Proof for C. (See Fig. 1.) From equations (6) and (7) follows

$$F_{x'}X + F_{y'}Y = 1, \tag{8}$$

as the equation for the tangent at a point $\theta = \theta_0$ to the Indicatrix for the point $(x_0,\,y_0)$, when X and Y are running coordinates, and the arguments of $F_{x'}$ and $F_{y'}$ are $x_0,\,y_0,\,\cos\theta_0,\,\sin\theta_0$.

For the perpendicular from a point $\bar{Q}(\bar{\theta})$ of the Indicatrix on the tangent at a point $Q(\theta)$, we find:

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*See Bolza, loc. cit., p. 120.

$$\bar{Q}M = \frac{-E\,(x_0,\ y_0;\ \cos\theta,\ \sin\theta;\ \cos\bar{\delta},\ \sin\bar{\delta})}{F(x_0,\ y_0,\ \cos\bar{\theta},\ \sin\theta)\ \sqrt{(F^2_{x'}+F^2_{y'})}},$$

while for the perpendicular from the origin G on the same tangent, we obtain

$$GM_G = \frac{-1}{\sqrt{(F^2_{x'}+F^2_{y'})}} < 0.$$

From the usual convention concerning the sign of a perpendicular, we obtain then:

$\bar{Q}M<0$, if $\bar{Q}$ and G are on the same side of QM,

$\bar{Q}M>0$, if $\bar{Q}$ and G are on opposite sides of QM,

from which the theorem is at once evident.

D. If the indicatrix for a point $(x_0,\ y_0)$ admits a double-tangent, touching the curve at the points $A(\alpha)$ and $A'(\alpha')$, the point $(x_0,\ y_0)$ is the corner of a possible discontinuous solution, the direction of the branches being α and α'.

Proof for D. From the equation for the tangent to the Indicatrix, (8), and from No. (V) of the general theorems (p. 121), we conclude that if the corner-condition is satisfied at a point (x_0, y_0) for two directions α and α', the tangents at the points $A(\alpha)$ and $A(\alpha)$ to the Indicatrix for the point $(x_0,\ y_0)$ must be coincident, From this, the theorem follows immediately.

After these remarks, we can now go over to the subject proper of this paper, the treatment of a simple problem of the Calculus of Variations, making use of the Indicatrix. The problem treated was given by Caratheadory himself.*

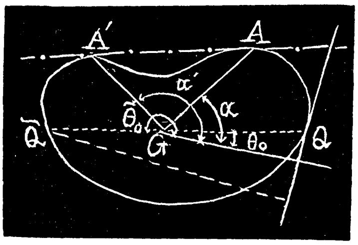

Fig. 1.

3. It is required to minimize the definite integral:

$$I = \int_{t_1}^{t_2} \left[\frac{\sqrt{[x'^2(y^2+1) - 2xyx'y' + y'^2(x^2+1)]}}{x^2+y^2+1} - \frac{\sqrt{(x'^2+y'^2)}}{4} \right] dt.$$

We have then:

$$F(x,\ y,\ x',\ y') = \frac{\sqrt{[x'^2(y^2+1) - 2xyx'y' + y'^2(x^2+1)]}}{x^2+y^2+1} - \frac{\sqrt{(x'^2+y'^2)}}{4},$$

and find

$$F_{x'y} = F_{xy'}\ ,\quad F_1 = \frac{1}{[1+(xy'-x'y)^2]^{\frac{3}{2}}} - \tfrac{1}{4}.$$

*C. Caratheodory, Ueber die disc. Loesungen, etc., p. 38.

(a) Euler's differential equation becomes

$$(x'y''-x''y')F=0.$$

The general integral of this is $x=my+n$, which represents the straight lines of the plane. We obtain also a singular solution from $F_1=0$,

$$1+(xy'-x'y)^2=4^{2/3}, \quad xy'+x'y=(4^{2/3}-1)^{\frac{1}{2}}.$$

We choose now the arc length s, as our functional parameter, and make the following transformation of coordinates (see Fig. 2):

$$\left.\begin{matrix} x=r\cos\phi, & x'=\cos\theta \\ y=r\sin\phi, & y'=\sin\theta \\ \psi=\theta-\phi & \end{matrix}\right\}. \tag{9}$$

The singular integral becomes then

$$r\sin\psi=(4^{2/3}-1)^{\frac{1}{2}},$$

also representing a straight line.

The solutions of Euler's differential equation called *extremals* prove to be the straight lines of the plane.

In the sequel, we denote by a, the constant $\sqrt{(4^{2/3}-1)}$.

Applying the transformations (9) to the functions $F(x, y, x', y')$, and $F_1(x, y, x', y')$, we get

$$F=\frac{\sqrt{[1+r^2\sin^2\psi]}}{r^2+1}-\tfrac{1}{4},$$

$$F_1=\frac{1}{[1+r^2\sin^2\psi]^{\frac{3}{2}}}-\tfrac{1}{4}. \tag{10}$$

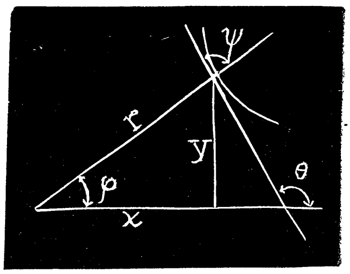

Fig. 2.

(b) For Legendre's condition, we have to consider the sign of F_1. We find from (10 a):

$$F_1>0, \text{ if } r\sin\psi<a, \quad F_1<0, \text{ if } r\sin\psi>a.$$

It follows, that the straight lines, which intersect the circle of radius a (denoted by C_a, see Fig. 3, are minima, and those lying outside C_a are maxima.

(c) The extremals being straight lines, it follows from the geometrical interpretation of the conjugate point,* that Jacobi's condition is fulfilled by every straight line of the plane.

*See Bolza, loc. cit., pp. 60-63, p. 137.

We have then the following result:

I. *Every straight line in the plane intersecting C_a furnishes at least a weak minimum.*

Every straight line in the plane, lying outside C_a, furnishes at least a weak maximum.

The straight lines which are tangent to C_a form a limiting case which will be considered later.

(d) We find:

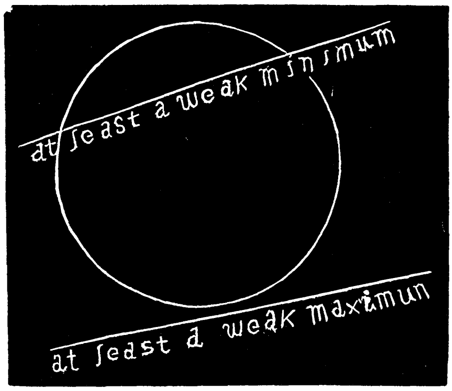

Fig. 3.

$$E(x,\ y;\ \cos\theta,\ \sin\theta;\ \cos\bar{\theta},\ \sin\bar{\theta})=$$

$$=\frac{1+r^2\sin^2\bar{\psi}}{\sqrt{[1+r^2\sin^2\bar{\psi}]\ (r^2+1)}}-\frac{\cos(\psi-\bar{\psi})+r^2\sin.\psi\ \sin.\bar{\psi}}{\sqrt{[1+r^2\sin^2\psi]\,(r^2+1)}}+\frac{\cos(\psi-\bar{\psi})-1}{4},$$

where (7) is applied and $\bar{\psi}=\bar{\theta}-\phi$.

For the further investigation, we have to discuss the sign of this function, which is a very cumbersome problem. At this point, we introduce the Indicatrix for this problem, by means of which the remaining questions can be more readily answered.

4. The *Indicatrix* is determined by the equation:

$$\rho=\frac{1}{\sqrt{\dfrac{(1+r^2\sin^2\psi)}{r^2+1}}-\frac{1}{4}}=\frac{1}{\mathrm{F}(r,\ \psi)}. \tag{11}$$

$r\equiv\sqrt{(x^2+y^2)}$ functions here as a parameter, which takes all real positive values, thus furnishing a curve for every point of the plane.

For the character of the Indicatrix it is of importance to know the sign of $\mathrm{F}(r,\ \psi)$ and $\mathrm{F}_1(r,\ \psi)$ for all values of ψ between 0 and 2π.* Since $\frac{\partial \mathrm{F}}{\partial\psi}>0$, for $0\leq\psi\leq\frac{1}{2}\pi$, we have:

$$\mathrm{F}_{\text{min.}}=\mathrm{F}(r,\ 0)=\frac{1}{1+r^2}-\tfrac{1}{4},$$

$$\mathrm{F}_{\text{max.}}=\mathrm{F}(r,\ \tfrac{1}{2}\pi)=\frac{1}{\sqrt{(1+r^2)}}-\tfrac{1}{4}.$$

We conclude:

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*See page 121, Theorems *A* and *B*.

F *constantly positive,* if $F_{\text{min.}}>0$, *i. e.*, if $\dfrac{1}{1+r^2}>\frac{1}{4}$ or $r<\sqrt{3}$.

F *constantly negative,* if $F_{\text{max.}}<0$, *i. e.*, if $\dfrac{1}{\sqrt{(1+r^2)}}<\frac{1}{4}$ or $r<\sqrt{15}$.

F *varying,* if $F_{\text{min.}}<0$, $F_{\text{max.}}>0$, *i. e.*, if $\sqrt{3}<r<\sqrt{15}$.*

We have previously found, that:

$$F_1>0, \text{ if } r\sin\varphi<a.$$
$$F_1<0, \text{ if } r\sin\varphi>a.$$

These results show, that the character of the Indicatrix will be essentially different for points lying in one of the four regions, into which the plane is divided by the three circles of radius a, $\sqrt{3}$, and $\sqrt{15}$, respectively (see Fig. 4, circles C_a, C_3, and C_{13}).

The problem is now reduced to the discussion of the properties of the Indicatrix in each of the four cases:

I. $0<r<a$.
II. $a<r<\sqrt{3}$.
III. $\sqrt{3}<r<\sqrt{15}$.
IV. $\sqrt{15}<r$,

after which the limiting cases $r=a$, $\sqrt{3}$, and $\sqrt{15}$, respectively, still have to be considered.

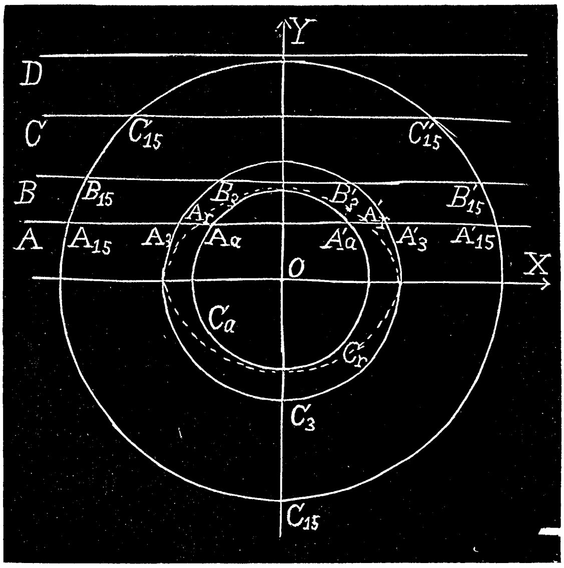

Fig. 4.

(To be continued.)

*$r=\sqrt{3}$ and $r=\sqrt{15}$ are two limiting cases, which will be considered later.

182. Proposed by O. L. CALLICOTT, Gettysburg, S. Dak.

Find the value of $\sqrt{2}\,\sqrt[3]{2}\,\sqrt[4]{2}\,\sqrt[5]{2}\ldots\sqrt[1000]{2}$.

I. Solution by W. D. LAMBERT, Washington, D. C.

Let $\frac{1}{2}P$ be the required product.

Then $\log_{10}P=\log_{10}2(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\ldots\frac{1}{1000})$.

By the Euler-Bernoulli formula for reducing summation to integration,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{x}=v+\log_e x+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}\cdots$$

v is Euler's constant$=0.57721,\ 56649\ldots$

$\therefore \log_{10}P=0.3010300(0.57721566+6.90775528+0.00050000-0.00000008\ldots)$ $=2.253351$.

$\therefore \frac{1}{2}P=89.603$, the required value.

II. Solution by S. A. COREY, Hiteman, Iowa.

If s is the value sought,

$$\log s=(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{1000})\log 2. \tag{1}$$

If $\log(1+x)$ be developed by the formula given in Prize Problem 237, Calculus, we get

$$\log(1+x)=0+\frac{x}{2m}\left[\frac{1}{1+x}+1+2\left(\frac{1}{1+\frac{x}{m}}+\frac{1}{1+\frac{2x}{m}}+\ldots+\frac{1}{1+\frac{(m-1)x}{m}}\right)\right]$$

$$+\frac{B_1x^2}{m^2.2!}\left[\frac{1}{(1+x)^2}-1\right]-\frac{B_2x^4}{m^4.4!}\left[\frac{3!}{(1+x)^4}-3!\right]+\text{ etc.} \tag{2}$$

If, now, x is an integer and m be taken equal to x, we have, as x approaches ∞,

$$\lim_{n\doteq\infty}\log n-\left[\tfrac{1}{2}+\sum_{r=2}^{r=\infty}r^{-1}\right]=-\left[\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}\cdots\right]=-b; \tag{3}$$

whence if C be Euler's constant, .577,215,664,901,5..., $b=C-\frac{1}{2}$. Substituting in (2), transposing, adding $\frac{1}{1000}$ to each member, and reducing, we get, by making $(1+x)=1000$,

$$(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\tfrac{1}{1000})=\log1000-1+\tfrac{1}{2000}+C-\left[\frac{B_1}{2.1000^2}-\frac{B_2}{4.1000^4}+\cdots\right]$$

$=6.485,470,860,55$, or $s=2^{6.485,470,860,55}=89.602,734.8\ldots$

Also solved by G. B. M. Zerr, J. Scheffer, and A. H. Holmes.

GEOMETRY.

312. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania, Philadelphia, Pa.

A variable circle passes through a fixed point and is tangent to a given circle. If a diameter of the first circle passes through the fixed point find the locus of its other extremity.

Solution by G. W. GREENWOOD, M. A. (Oxon), Roanoke College, Salem, Va.

Since the circle of a radius vector of a central conic is always tangent to its auxiliary circle, the required locus is the conic having the given circle as auxiliary circle and the given point as one focus.

Or, the following method:

Call the center of the given circle 0; its radius, a; the given point, S; the point symmetrical to S with respect to 0, S'; the radius of the circle through S tangent to the given circle, r; the other extremity of the diameter through S, P; the mid-point of SP (the center of the tangent circle), A. Suppose that S is *without* the circle. If the circles are tangent externally, we have

$$PS'-PS=2(AO-AS)=2(AO-r)=2a.$$

If the circles are tangent internally, we get

$$PS-PS'=2a.$$

Therefore, the locus of P is an hyperbola.

Next, suppose that S is *within* the given circle. We get

$$PS'+PS=2(AO+r)=2a.$$

Therefore, the locus of P is an ellipse.

In either case the conic has the points S and S' as foci.

The cases in which S coincides with the center of the given circle, or is on the given circle, may easily be dealt with independently.

Also solved by G. B. M. Zerr, J. Scheffer, C. N. Schmall, and J. S. Brown.

313. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that an algebraic curve of odd degree which is symmetrical with respect to a center has the center on the curve.

No solution has been received.

314. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Find the area of the triangle bounded by the lines $l\alpha+m\beta+n\gamma=0$; $l'\alpha+m'\beta+n'\gamma=0$; $l''\alpha+m''\beta+n''\gamma=0$, where α stands for $x\cos\alpha+y\sin\alpha-p$, etc. [See Salmon's *Conic Sections*, 6th ed., p. 130, Ex. 1.]

Solution by the PROPOSER.

The formula for the double area of the triangle whose sides are the lines $Ax+By+C=0$, $A'x+B'y+C'=0$, and $A''x+B''y+C''=0$, is

$$\frac{\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}^2}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix} \times \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix} \times \begin{vmatrix} A'' & B'' \\ A & B \end{vmatrix}}$$

(V. Salmon's *Conic Sections*, §39.)

For $2\triangle$, the double area of the triangle of reference, the lengths of whose sides are represented by a, b, and c, the formula gives:

$$\frac{\begin{vmatrix} \cos\alpha & \sin\alpha & p \\ \cos\beta & \sin\beta & p' \\ \cos\gamma & \sin\gamma & p'' \end{vmatrix}^2}{\begin{vmatrix} \cos\alpha & \sin\alpha \\ \cos\beta & \sin\beta \end{vmatrix} \times \begin{vmatrix} \cos\beta & \sin\beta \\ \cos\gamma & \sin\gamma \end{vmatrix} \times \begin{vmatrix} \cos\gamma & \sin\gamma \\ \cos\alpha & \sin\alpha \end{vmatrix}}$$

$$\therefore \begin{vmatrix} \cos\alpha & \sin\alpha & p \\ \cos\beta & \sin\beta & p' \\ \cos\gamma & \sin\gamma & p'' \end{vmatrix}^2 = 2\triangle \sin A \sin B \sin C = \frac{2\triangle abc}{8R^3};$$

where $2R = \dfrac{a}{\sin A}$ = diameter of circle circumscribed about triangle of reference. For our problem the double area is

$$\frac{\begin{vmatrix} l\cos\alpha+m\cos\beta+n\cos\gamma & l\sin\alpha+m\sin\beta+n\sin\gamma & lp+mp'+np'' \\ l'\cos\alpha+m'\cos\beta+n'\cos\gamma & l'\sin\alpha+m'\sin\beta+n'\sin\gamma & l'p+m'p'+n'p'' \\ l''\cos\alpha+m''\cos\beta+n''\cos\gamma & l''\sin\alpha+m''\sin\beta+n''\sin\gamma & l''p+m''p'+n''p'' \end{vmatrix}^2}{\begin{vmatrix} l\cos\alpha+m\cos\beta+n\cos\gamma & l\sin\alpha+m\sin\beta+n\sin\gamma \\ l'\cos\alpha+m'\cos\beta+n'\cos\gamma & l'\sin\alpha+m'\sin\beta+n'\sin\gamma \end{vmatrix} \times}$$

$$\times \begin{vmatrix} l'\cos\alpha+\dots & l'\sin\alpha+\dots \\ l''\cos\alpha+\dots & l''\sin\alpha+\dots \end{vmatrix} \times \begin{vmatrix} l''\cos\alpha+\dots & l''\sin\alpha+\dots \\ l\cos\alpha+\dots & l\sin\alpha+\dots \end{vmatrix}$$

$$= \frac{\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ p & p' & p'' \end{vmatrix}^2 \times \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}^2}{\dfrac{1}{2R}\begin{vmatrix} a & b & c \\ l & m & n \\ l' & m' & n' \end{vmatrix} \times \dfrac{1}{2R}\begin{vmatrix} a & b & c \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \times \dfrac{1}{2R}\begin{vmatrix} a & b & c \\ l'' & m'' & n'' \\ l & m & n \end{vmatrix}}$$

And the required result is
$$\frac{\triangle abc \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}^2}{\begin{vmatrix} a & b & c \\ l & m & n \\ l' & m' & n' \end{vmatrix} \times \begin{vmatrix} a & b & c \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \times \begin{vmatrix} a & b & c \\ l'' & m'' & n'' \\ l & m & n \end{vmatrix}}$$

Also solved by G. B. M. Zerr, and C. N. Schmall.

CALCULUS.

237. *Prize Problem. Proposed by S. A. COREY, Hiteman, Iowa.

Find an expression for the remainder after n terms in the following development† of $f(a+x)$:

$$f(a+x)=f(a)+\frac{x}{m.2}\Big\{f'(a+x)+f'(a)+2\Big[f'\Big(a+\frac{x}{m}\Big)+f'\Big(a+\frac{2x}{m}\Big)$$
$$+f'\Big(a+\frac{3x}{m}\Big)+\ldots+f'\Big(a+\frac{m-1}{m}x\Big)\Big]\Big\}-\frac{B_1x^2}{m^2.2!}\Big[f''(a+x)-f''(a)\Big]$$
$$+\frac{B_2x^4}{m^4.4!}\Big[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\Big]-\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}\Big[f^{(2n)}(a+x)-f^{(2n)}(a)\Big]$$

$+\ldots$, B_1, B_2, ... being Bernoulli's numbers.

Solution by the PROPOSER.

Taylor's development for $f(a+x)$ may be written,

$$0=f(a)-f(a+x)+xf'(a)+\frac{x^2}{2!}f''(a)+\frac{x^3}{3!}f'''(a)+\ldots+\frac{x^n}{n!}f^{(n)}(a+\theta_1x), \qquad (1)$$

where $0<\theta_1<1$.

If $f(b-x)$ be similarly developed, and if b be made equal to $(a+x)$ *after* differentiation, we get,

$$0=f(a+x)-f(a)-xf'(a+x)+\frac{x^2}{2!}f''(a+x)-\frac{x^3}{3!}f'''(a+x)+\ldots$$
$$+(-1)^nf^{(n)}(a+\theta_2x), \qquad (2)$$

where $0<\theta_2<1$.

Adding (1) and (2),

$$0=-x[f'(a+x)-f'(a)]+\frac{x^2}{2!}[f''(a+x)+f''(a)]-\ldots$$
$$-\frac{x^n}{n!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \qquad (3)$$

(n odd).

Dividing (3) by x and letting $z_n=[f^{(n)}(a+x)+f^{(n)}(a)]$ and $y_n=[f^{(n)}(a+x)-f^{(n)}(a)]$, we get, by lowering by unity the order of each derivative,

*In order to emphasize the importance of finding such an expression for the remainder after n terms as will hold good for all integral values of m and approach 0 as m approaches ∞, and at the same time enable computers to determine absolutely in what cases the development holds, the proposer offers a prize of $10 for the best solution. ED. F.

†See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

$$0=-y_0+\frac{x}{2!}z_1-\frac{x^2}{3!}y_2+\frac{x^3}{4!}z_3-\frac{x^4}{5!}y_5+\dots$$

$$-\frac{x^n}{(n+1)!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \qquad (4)$$

(n even).

Now develop $f''(a+x)$ by (4), multiply the result by $-(2x^2/4!)$, and add to (4). In the resulting sum, (5), the term involving z_3 has been eliminated.

$$0=-y_0+\frac{x}{2}z_1-\frac{B_1x^2}{2!}y_2+\tfrac{2}{15}\cdot\frac{x^4}{4!}y_4-\tfrac{1}{4}\cdot\frac{x^5}{5!}z_5+\tfrac{5}{14}\cdot\frac{x^6}{6!}y_6-\tfrac{11}{24}\cdot\frac{x^7}{7!}z_7+\tfrac{5}{9}\cdot\frac{x^8}{8!}y_8$$

$$-\tfrac{13}{20}\cdot\frac{x^9}{9!}z_9+\dots+\tfrac{1}{12}\cdot\frac{x^n}{(n-1)!}[f^{(n)}(a+\theta_4x)-f^{(n)}(a+\theta_3x)]$$

$$-\frac{x^n}{(n+1)!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \qquad (5)$$

B_1 being Bernoulli's first number, $\frac{1}{6}$.

In a similar manner the term in (5) containing z_5 may be eliminated by developing $f^{\text{iv}}(a+x)$ by (4), multiplying result by $+\frac{3x^4}{6!}$ and adding to (5). By continuing this process we may eliminate as many z's as we please, except z_1. Thus to eliminate z_3, z_5, z_7, z_9, and z_{11}, successively, the necessary multipliers are, respectively,

$$-\frac{2x^2}{4!},\quad +\frac{3x^4}{6!},\quad -\tfrac{20}{3}\cdot\frac{x^6}{8!},\quad +\frac{21x^8}{10!},\ \text{and}\ -\frac{90x^{10}}{12!}.$$

The result obtained by eliminating these five z's is found to be identical with the formula given in the problem for the case $m=1$, up to and including the term containing B_5, and may be thus written,

$$0=-y_0+\frac{x}{2!}z_1-\frac{B_1x^2}{2!}y_2+\frac{B_2x^4}{4!}y_4-\frac{B_3x^6}{6!}y_6$$

$$+\frac{B_4x^8}{8!}y_8-\frac{B_5x^{10}}{10!}y_{10}+\frac{x^{12}}{12!}R_{12} \qquad (6)$$

$$\text{where } R_{12}=-12!\left(\frac{\varepsilon_1}{13!}-\frac{2\varepsilon_2}{4!\,11!}+\frac{3\varepsilon_3}{6!\,9!}-\frac{20\varepsilon_4}{3.8!\,7!}+\frac{21\varepsilon_5}{10!\,5!}-\frac{90\varepsilon_6}{12!\,3!}\right), \qquad (7)$$

the ε's having some value between the greatest and least value of $[f^{12}(a+\theta_1x)-f^{12}(a+\theta_2x)]$.

Let r_{12} be positive and equal to the greatest numerical value which ε may take. Then as

$$12!\left(\frac{1}{13!}+\frac{2}{4!\ 11!}+\frac{3}{6!\ 9!}+\frac{20}{3.8!\ 7!}+\frac{21}{10!\ 5!}+\frac{90}{12!\ 3!}\right)r_{12}=\tfrac{27478}{455}r_{12},$$

$$-\tfrac{27478}{455}r_{12}<R_{12}<+\tfrac{27478}{455}r_{12},$$

where the coefficient of r_{12} is obtained by changing all the negative signs in (7) to positive, substituting unity for each of the ϵ's and adding.

If now

$$f\left(a+\frac{x}{m}\right),\ f\left[\left(a+\frac{x}{m}\right)+\frac{x}{m}\right],\ f\left[\left(a+\frac{2x}{m}\right)+\frac{x}{m}\right],\ \ldots,\ f\left[\left(a+\frac{m-1}{m}x\right)+\frac{x}{m}\right],$$

be developed successively by (6), simplifying each by the aid of the preceding, the result is identical with the formula given in the problem up to and including B_5, and may be thus written,

$$f(a+x)=f(a)+\frac{x}{m.2}\left[f'(a+x)+f'(a)+2\left(f'(a+\frac{x}{m})+f'(a+\frac{2x}{m})+\ldots\right.\right.$$

$$\left.\left.+f'(a+\frac{m-1}{m}x)\right)\right]-\frac{B_1x^2}{m^2.2!}[f''(a+x)-f''(a)]+\frac{B_2x^4}{m^4.4!}[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)]$$

$$-\frac{B_3x^6}{m^6.6!}[f^{\text{vi}}(a+x)-f^{\text{vi}}(a)]+\frac{B_4x^8}{m^8.8!}[f^{\text{viii}}(a+x)-f^{\text{viii}}(a)]$$

$$-\frac{B_5x^{10}}{m^{10}.10!}[f^{\text{x}}(a+x)-f^{\text{x}}(a)]\pm\frac{x^{12}.m.R_{12}}{m^{12}.12!}. \qquad (8)$$

By a close inspection of the method used to eliminate the z_3, z_5, z_7, z_9, and z_{11} from (4), we learn that the following formula may be generally used to obtain the multiplier, $\frac{c_s x^{(s-1)}}{(s+1)!}$, which may be employed to eliminate z_s after all the preceding z's except z_1 have been eliminated, viz.,

$$0=S!\left[\frac{1}{(s+1)!}-\frac{2}{4!\ (s-1)!}+\frac{3}{6!\ (s-3)!}-\tfrac{20}{3}\cdot\frac{1}{8!\ (s-5)!}\right.$$

$$\left.+\frac{21}{10!\ (s-7)!}-\frac{90}{12!\ (s-9)!}+\ldots+\frac{c_s}{(s+1)!2}\right]. \qquad (9)$$

which must hold for *all odd* values of s except $s=1$. It can also be shown that when s is even, say $2n$, (9) gives the value of $\pm B_n$ according as n is odd or even.

After thus eliminating z_3, z_5, z_7, z_9, ... and z_s from (4), and repeating the process used to obtain (8), we obtain a development which is identical with the development up to and including the term containing B_n where $n=\frac{s-1}{2}$. That this is true for *all* values of n follows from the fact that the two developments become identical when $R=0$ for *any* value of n.

Likewise by a method similar to that used to obtain R_{12}, we may obtain the value of $R_{(2n+2)}$. That method is: Find the sum of all the terms in (9) after making all negative signs positive, and after making $s=(2n+2)$, observing that *no new term is to be added.* Calling this sum $t_{(2n+2)}$ we may write

$$R_{(2n+2)}=\pm t_{(2n+2)}\,[f^{(2n+2)}\,(a+\theta_1 x)-f^{(2n+2)}\,(a+\theta_2 x)],$$

where both θ_1 and θ_2 have some value greater than zero and less than unity, but not the same value unless $R=0$.

The remainder after the term involving B_n may therefore be expressed as follows:

$$\pm\frac{x^{(2n+2)}\,.\,t_{(2n+2)}}{m^{(2n+1)}\,.\,(2n+2)\,!}[f^{(2n+2)}\,(a+\theta_1 x)-f^{(2n+2)}\,(a+\theta_2 x],^*$$

where the value of $t_{(2n+2)}$ must be computed for higher values of n than are here given, viz.,

$$t_4=\tfrac{8}{15},\ t_6=\tfrac{8}{7},\ t_8=\tfrac{148}{45},\ t_{10}=\tfrac{410}{33},\ t_{12}=\tfrac{27478}{455},\ t_{14}=\tfrac{3340}{9},\ \text{and}\ t_{16}=\tfrac{241048}{85}.$$

To obtain such higher values of t it is, of course, necessary first to find values of c_s in (9) higher than the following, which were used to find the foregoing values of t, viz.,

$$c_3=-2,\ c_5=+3,\ c_7=-\tfrac{20}{3},\ c_9=+21,\ c_{11}=-90,\ c_{13}=+\tfrac{7601}{15},\ \text{and}\ c_{15}=-3640.$$

The foregoing proof is based on the assumption that all the derivatives are finite and continuous between the limits $x=0$ and $x=x$.

It will be observed that the foregoing solution does not give the value of the remainder after the *general* term as required in the problem, the value of t not being given for more than the first few terms. In actual work the values of t here given are probably all that the computer needs. But, if not, higher values may be found without excessive labor in the manner given. This solution, therefore, places the formula given in the problem on a fairly satisfactory basis in so far as the needs of the computer are involved.

As the Method of Approximation which the writer gave in the June-July, 1906, number of the MONTHLY is based on the formula of the problem,

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*The quantitity within the brackets cannot exceed twice the *greatest* variation of $f^{(2n+2)}(a+\theta x)$ for *any single interval* $(x\ m)$ as x increases from $x=0$ to $x=x$.

this solution will serve to determine the limits within which errors in results due to the dropping of higher terms must lie whenever it is possible to determine the limits within which the value of the lowest dropped derivative lies.

NOTE. Mr. Corey informs us that he did not succeed in obtaining a solution until a month after the problem was published, though he had repeatedly attempted to do so. Mr. Corey's solution is quite satisfactory so far as its use for practical computation is concerned, yet greater value would result were an expression found for the remainder after the general term. A prize of $10 is still offered by Mr. Corey for finding an expression for the remainder after the nth term. ED. F.

238. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equations

$$(a)\quad (x+y^2)dx+(x^2+y)dy=0,$$

$$(b)\quad (x+xy+y^2)dx-(x^2+xy-y)dy=0.$$

Solution by GEORGE W. HARTWELL, Columbia University, New York.

$(a)\quad (x+y^2)dx+(x^2+y)dy=0.$

This equation can be written

$$\frac{dx}{x^2+y}=\frac{dy}{-x-y^2};\ \therefore\frac{dx+dy}{(x-y)\ (x+y-1)}=\frac{xdy+ydx+dx+dy}{(x-y)(xy-1)}.$$

Multiplying both members by $x-y$,

$$\frac{dx+dy}{x+y-1}=\frac{xdy+ydx+dx+dy}{xy-1}.$$

Clearing of fractions and transposing,

$$(x+y-1)(xdy+ydx)-(xy-1)(dx+dy)+(x+y-1)(dx+dy)=0.$$

Dividing by $(x+y-1)^2$,

$$\frac{(x+y-1)\ (xdy+ydx)-(xy-1)\ (dx+dy)}{(x+y-1)^2}+\frac{dx+dy}{x+y-1}=0.$$

Integrating, $\dfrac{xy-1}{x+y-1}+\log(x+y-1)=a$. Multiplying by $(x+y-1)$,

$$xy-1+(x+y-1)\log(x+y-1)=a(x+y-1).$$

$(b)\quad (x+xy+y^2)dx-(x^2+xy-y)dy=0.$

This equation can be written

$$\frac{dx}{x^2+xy-y}=\frac{dy}{y^2+xy+x};\ \therefore\frac{dx-dy}{(x+y)\ (x-y-1)}=\frac{xdx+ydy}{(x+y)\ (x^2+y^2)}.$$

Multiplying by $(x+y)$, $\dfrac{dx-dy}{x-y-1}=\dfrac{xdx+ydy}{x^2+y^2}$. Integrating,

$$2\log(x-y-1)=\log(x^2+y^2)+\log a;\ \therefore (x-y-1)^2=a(x^2+y^2).$$

Also solved by G. B. M. Zerr.

MECHANICS.

198. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Three spheres of the same material, radii R, r, S, rest upon a horizontal plane, touching each other. Find the radius of a sphere of the same material as the others which, being placed upon the other three spheres, will just prevent the latter from separating, the coefficient of friction between the spheres being μ, and between the spheres and the table being μ'

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x=radius of top sphere center Q, R=radius of bottom sphere center P. The forces acting on the sphere radius R are its own weight W, the friction F, the friction F', and the normal reaction between the two spheres R. Let nW_1 be the part of the weight of the sphere radius x supported by the sphere radius R. $\angle DPC=\beta$.

Now $FHC=RDC$. $F=R\left(\dfrac{DC}{HC}\right)=R\tan\frac{1}{2}\beta$. $\therefore F=\mu R$ or $\mu=\tan\frac{1}{2}\beta$.

$F'=\mu'[W+nW_1]=R\sin\beta-F\cos\beta=R(\sin\beta-\tan\frac{1}{2}\beta\cos\beta]=R\tan\frac{1}{2}\beta$.

$$\therefore R=\frac{\mu'[W+nW_1]}{\tan\frac{1}{2}\beta}=\frac{\mu'}{\mu}[W+nW_1].$$

Resolving vertically, $nW_1=R\cos\beta+F\sin\beta$.

$\therefore nW_1=R[\cos\beta+\tan\frac{1}{2}\beta\sin\beta]=R$. $\therefore nW_1=\dfrac{\mu'}{\mu}[W+nW_1]$.

Let δ=density of each sphere. $\therefore W=\delta g\times\frac{4}{3}\pi R^3$, $W_1=\delta g\times\frac{4}{3}\pi x^3$.

$\therefore nx^3=\dfrac{\mu'}{\mu}[R^3+nx^3]$. $\therefore n=\dfrac{\mu' R^3}{[\mu-\mu']x^3}$.

Let mW_1 be the part of the upper sphere supported by sphere radius r, and pW_1 the part supported by sphere radius S. Then

$$m=\frac{\mu' r^3}{[\mu-\mu']x^3},\ p=\frac{\mu' S^3}{[\mu-\mu']x^3}.$$

But $n+m+p=1$.

$$\therefore \frac{\mu'[R^3+r^3+S^3]}{\mu-\mu'}=x^3, \text{ and } \therefore x=\sqrt[3]{\frac{\mu'[R^3+r^3+S^3]}{\mu-\mu'}}.$$

199. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

A sphere of water, radius $\frac{1}{49}$, the earth's radius, is brought together by mutual attractions of particles from a state of infinite diffusion. Find its temperature owing to the amount of work done by these forces.

Solution by W. D. LAMBERT, Washington, D. C.

Let r be the variable radius of the sphere, while forming; a the final radius $=\frac{6.370\times10^8}{49}$ centimeters; ρ the density of water$=1$; k the gravitation constant$=6.665\times10^{-8}$ dynes; J the mechanical equivalent of heat$=4.184\times10^7$ ergs for the centigrade gram-calorie. It is a little easier to conceive the sphere as pulled asunder against its own attraction, and the amount of work will be the same. Suppose the sphere made up of layers, each of thickness dr, and that the sphere has been reduced to radius r. The mass of a layer is $4\pi\rho r^2dr$. The attraction between this mass and the remaining sphere is $\frac{k\times\frac{4}{3}\pi\rho r^3\times4\pi\rho r^2dr}{x^2}$, where x denotes the distance of the layer (supposed to be scattered symmetrically) from the center of the sphere. The work done in removing the layer from the surface of the sphere in question to infinity is $\frac{16}{3}\pi^2k\rho^2r^5dr\int_r^\infty\frac{dx}{x^2}=\frac{16}{3}\pi^2k\rho^2r^4$.

The total work done in removing all layers is

$$\frac{16}{3}\pi^2k\rho^2\int_0^a r^4dr=\frac{16}{15}\pi^2k\rho^2a^5.$$

Dividing this quantity by the mass and by J we get for the tempera- $\frac{4}{5}\frac{\pi k\rho a^2}{J}$. For substances other than water this result should be multiplied by the specific heat of the substance. Using the numerical values previously given, we get for the temperature $0^\circ.677$ centigrade.

Also solved by G. B. M. Zerr, whose result is 0.656. This difference of result is due to the different values assumed for the constants entering into the solution.

AVERAGE AND PROBABILITY.

183. **Proposed by J. EDWARD SANDERS, Reinersville, Ohio.**

A point within a given triangle is joined to each of the corners. What is the average of the sum of the lengths of these three lines?

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.**

Let ABC be the given triangle, P the random point, A the vertex, BC the base of the triangle, AD the altitude. Through P draw QR, parallel to BC cutting AD in F. Let $AD=p$, $BD=e$, $DC=d$, $AF=x$, $FP=y$. Then $AP=\sqrt{[x^2+y^2]}$. The limits of x are 0 and p; of y, $-QF=ex/p$ and $+FR=dx/p$. Let M=average length of AP, Δ=average length of the sum.

$$\therefore M=\int_0^p\int_{-ex/p}^{dx/p}\sqrt{[x^2+y^2]}\,dxdy\Big/\int_0^p\int_{-ex/p}^{dx/p}dxdy$$

$$=\frac{2}{ap}\int_0^p\int_{-ex/p}^{dx/p}\sqrt{[x^2+y^2]}dxdy, \text{ since } d+e=a,$$

$$=\frac{1}{ap}\int_0^p\left[\frac{[bd+ce]x^2}{p^2}+x^2\log\left(\frac{d+b}{c-e}\right)\right]dx, \ \left(\begin{matrix}p^2+d^2=b^2\\ p^2+e^2=c^2\end{matrix}\right.$$

$$=\frac{1}{3a}\left[bd+ce+p^2\log\left(\frac{d+b}{c-e}\right)\right]$$

$$=\frac{1}{3a}\left[b^2\cos C+c^2\cos B+b^2\sin^2C\log\left(\frac{b[1+\cos C]}{b[1-\cos B]}\right)\right].$$

By similarity,

$$\Delta=\frac{1}{3a}\left[b^2\cos C+c^2\cos B+\frac{4\Delta^2}{a^2}\log\left(\frac{a+b+c}{b+c-a}\right)\right]+\frac{1}{3b}\left[c^2\cos A+a^2\cos C\right.$$

$$\left.+\frac{4\Delta^2}{b^2}\log\left(\frac{a+b+c}{a+c-b}\right)\right]+\frac{1}{3c}\left(a^2\cos B+b^2\cos A+\frac{4\Delta^2}{c^2}\log\left(\frac{a+b+c}{a+b-c}\right)\right],$$

where Δ=area of triangle.

$$\therefore \Delta=\tfrac{1}{3}[a+b+c]+\frac{1}{6a^2}[b+c][b-c]^2+\frac{1}{6b^2}[a+c][a-c]^2$$

$$+\frac{1}{6c^2}[a+b][a-b]^2+\frac{4\Delta^2}{3}\left[\frac{1}{a^3}\log\left(\frac{a+b+c}{b+c-a}\right)+\frac{1}{b^3}\log\left(\frac{a+b+c}{a+c-b}\right)\right.$$

$$\left.+\frac{1}{c^2}\log\left(\frac{a+b+c}{a+b-c}\right)\right].$$

If $a=b=c$, $\Delta=a[1+\frac{3}{4}\log 3]$.

II. Solution by HENRY HEATON, Belfield, N. D.

Let P be the point, and $AD=h$, the perpendicular from A upon BC. Put $AP=x$, and $\angle PAD=\theta$. Then the average length of AP is

$$\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\int_0^{h\sec\theta}x^2d\theta dx\div\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\int_0^{h\sec\theta}xd\theta dx=\frac{2h}{3}\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\sec^3\theta d\theta\div\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\sec^2\theta d\theta$$

$$=\frac{h}{3}\Big(\cot C\operatorname{cosec}C+\cot B\operatorname{cosec}B-\log[\tan\tfrac{1}{2}C\tan\tfrac{1}{2}B]\div\cot C+\cot B$$

$$=\frac{1}{3a}\Big(b^2\cos C+c^2\cos B-bc\sin B\sin C\log[\tan\tfrac{1}{2}C\tan\tfrac{1}{2}B].$$

In like manner it may be shown that the average length of BP is

$$\frac{1}{3b}\Big(a^2\cos C+c^2\cos A-ac\sin A\sin C\log[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}C]\text{; and of } CP,$$

$$\frac{1}{3c}\Big(a^2\cos B+b^2\cos A-ab\sin A\sin B\log[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B].$$

Hence the required average is

$$M=\tfrac{1}{3}\Big[\Big(\frac{b^2}{c}+\frac{c^2}{b}\Big)\cos A+\Big(\frac{a^2}{c}+\frac{c^2}{a}\Big)\cos B+\Big(\frac{a^2}{b}+\frac{b^2}{a}\Big)\cos C-a\sin\tfrac{1}{2}A[b\sin B$$

$$+c\sin C]\log\tan\tfrac{1}{2}A-b\sin B[a\sin A+c\sin C]\log\tan\tfrac{1}{2}B$$

$$-c\sin C[a\sin A+b\sin B]\log\tan\tfrac{1}{2}C.$$

184. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of the sides of a given square, straight lines are drawn across the square in every possible direction. What is their average length?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The problem evidently wants the average length of all lines terminated in opposite sides; otherwise the problem is the same as problem 169, three solutions of which have already been published.

Let $[a, x]$ be the coordinates of one end of the line, $[0, y]$ the coordinates of the other end. $\triangle$=the average length required.

$$\therefore \triangle=\frac{\int_0^a\int_0^x\sqrt{\{a^2+[x-y]^2\}}dx\,dy}{\int_0^a\int_0^x dxdy},=\frac{2}{a^2}\int_0^a\int_0^x\sqrt{\{a^2+[x-y]^2\}}dxdy$$

$$=\frac{1}{a^2}\int_0^a\Big(x\sqrt{[a^2+x^2]}+a^2\log\frac{x+\sqrt{[a^2+x^2]}}{a}\Big)dx=a\{\tfrac{2}{3}[1-\sqrt{2}]+\log[1+\sqrt{2}]\}.$$

II. Solution by HENRY HEATON, Belfield, N. D.

Let P be a point in AB, PE a line perpendicular to AB, and PF a line drawn across the $\triangle PAD$. Put $AP=x$, and $\angle FPA=\theta$. Then supposing x constant, the average length of the lines drawn from P across the triangle PAD is

$$\int_0^{\tan^{-1}(a/x)}x\sec\theta\,d\theta\div\int_0^{\tan^{-1}(a/x)}d\theta=x\log\Big(\frac{\sqrt{[a^2+x^2]}+a}{x}\Big)\div\tan^{-1}[a/x].$$

Similarly the average length of lines drawn from P across the $\triangle PED$ is

$$a\log\left(\frac{\sqrt{[a^2+x^2]}+x}{a}\right)\div\tan^{-1}[x/a].$$

Hence the average length of lines drawn from P across the rectangle $PEDA$ is

$$\frac{2}{\pi}\left[x\log\left(\frac{\sqrt{[a^2+x^2]}+a}{x}\right)+a\log\left(\frac{\sqrt{[a^2+x^2]}+x}{a}\right)\right].$$

Hence the required average is

$$M=\frac{4}{a\pi}\int_0^a\left[x\log\left(\frac{\sqrt{[a^2+x^2]}+a}{x}\right)+a\log\left(\frac{\sqrt{[a^2+x^2]}+x}{a}\right)\right]dx$$

$$=\frac{6a}{\pi}\log[\sqrt{2}+1]-\frac{2}{\pi}\int_0^a\frac{xdx}{\sqrt{[a^2+x^2]}}=\frac{2a}{\pi}\{3\log\sqrt{[2+1]}-\sqrt{[2+1]}\}.$$

NOTE.—These solutions differ because both problems are stated in the indefinite form and the authors have assumed different laws of distribution. ED. F.

MISCELLANEOUS.

168. **Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.**

Sum to n terms, $\sin\alpha\sin\beta+\sin\alpha-\beta\sin\beta+\gamma+\sin\alpha-2\beta\sin\beta+2\gamma+\ldots$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The series intended is evidently

$$\sin\alpha\sin\beta+\sin[\alpha-\beta]\sin[\beta+\gamma]+\sin[\alpha-2\beta]\sin[\beta+2\gamma]+\ldots \text{ to } n \text{ terms.}$$

Since $2\sin\alpha\sin\beta=\cos[\alpha-\beta]-\cos[\alpha+\beta]$ we get, if S=the sum

$$S=\tfrac{1}{2}\{\cos[\alpha-\beta]+\cos[\alpha-2\beta-\gamma]+\cos[\alpha-3\beta-2\gamma]+\ldots \text{ to } n \text{ terms}\}$$
$$-\tfrac{1}{2}\{\cos[\alpha+\beta]+\cos[\alpha+\gamma]+\cos[\alpha-\beta+\gamma]+\ldots \text{ to } n \text{ terms}\}.$$

Let $\alpha+\beta=\theta$, $=[\beta+\gamma]=\phi$, $[\alpha+\beta]=\psi$, $-[\beta-\gamma]=\rho$.

$$\therefore S=\tfrac{1}{2}\{\cos\theta+\cos[\theta+\phi]+\cos[\theta+2\phi]+\ldots+ \text{ to } n \text{ terms}\}$$
$$-\tfrac{1}{2}\{\cos\psi+\cos[\psi+\rho]+\cos[\psi+2\rho]+\ldots+ \text{ to } n \text{ terms}\}.$$

$$\therefore\ S=\frac{\cos[\theta+\frac{n-1}{2}\phi]\sin\frac{n\phi}{2}}{2\sin\frac{\phi}{2}}-\frac{\cos[\psi+\frac{n-1}{2}\rho]\sin\frac{n\rho}{2}}{2\sin\frac{\rho}{2}}$$

$$=\frac{\cos\{\alpha-\beta-\frac{n-1}{2}[\beta+\gamma]\}\sin\frac{n[\beta+\gamma]}{2}}{2\sin\frac{\beta+\gamma}{2}}-\frac{\cos\{\alpha+\beta-\frac{n-1}{2}[\beta-\gamma]\}\sin\frac{n[\beta-\gamma]}{2}}{2\sin\frac{\beta-\gamma}{2}}.$$

Also solved by A. H. Holmes.

PROBLEMS FOR SOLUTION.

GEOMETRY.

319. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Given the radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

MECHANICS.

204. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A set of particles have coplanar motion due to mutual attractions. Each particle is now affected with a velocity V parallel to a fixed direction. How will this affect the angular momentum of the set about their centroid?

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

147. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

If $4n+3$ is prime, $2(1, 2, 3, \ldots, 4n)+1\equiv 0 \pmod{4n+3}$; and conversely. If $4n+3$ is prime, $(1, 2, 3, \ldots, 2n)^2-4\equiv 0 \pmod{4n+3}$; and conversely.

AVERAGE AND PROBABILITY.

190. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line is drawn at random across a regular $2n$-gon; what is the chance that it crosses parallel sides?

MISCELLANEOUS.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If ϕ and ψ are small angles, show that an approximate value of ϕ/ψ is

$$\tfrac{2}{3}\frac{\sin\phi}{\sin\psi}+\tfrac{1}{3}\frac{\tan\phi}{\tan\psi}-\tfrac{1}{180}(\phi^2-\psi^2)(9\phi^2-\psi^2).$$

NOTES AND NEWS.

The August-September number of THE AMERICAN MATHEMATICAL MONTHLY will not appear until September 28th.

Dr. A. L. Underhill, Instructor at Princeton University, has been appointed to an Instructorship in Mathematics at the University of Wisconsin.

Professor A. L. Rhoton, of Southwestern University, Jackson, Tenn., has been appointed Professor of Mathematics at Georgetown College, Georgetown, Kentucky.

Mr. W. V. Lovitt, who took the Master's degree at the University of Chicago in June, has been appointed to an Instructorship in Mathematics at the University of Washington, Seattle, Wash.

At a recent meeting of the regents of the University of West Virginia, Dr. John A. Eiesland, of the United States Naval Academy, was elected to the chair of mathematics in that institution.

Dr. Oswald Veblen, Preceptor in Princeton University, has been promoted to an Assistant Professorship in Mathematics. Mr. Veblen received his Doctor's degree at the University of Chicago in 1903.

Mr. G. D. Birkhoff received his Doctorate in Mathematics at the University of Chicago at the June Convocation. Dr. Birkhoff has been appointed Instructor in Mathematics at the University of Wisconsin.

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AN EXAMPLE OF THE INDICATRIX IN THE CALCULUS OF VARIATIONS.*

By ARNOLD DRESDEN, The University of Chicago.

(Continued.)

§5. *Case I:* $0<r<a$.

From our previous work, we obtain

$$F(r, \psi)>0, \quad F_1(r, \psi)>0, \quad 0\leqq\psi\leqq 2\pi.$$

We conclude from this, by theorems A and B (see page 121), that the Indicatrix for any point in this region is a curve, which is closed around the origin G_1 and which has constantly positive curvature (see Fig. 5, curve A). It follows, that all the tangents lie entirely outside the curve, so that all points $\bar{Q}$ lie on the same side of the tangent at *any* point Q, as the origin G. Since, moreover, $F>0$, we have by theorem C (see page 122), that

$$E(x, y; \cos\theta, \sin\theta; \cos\bar{\theta}, \sin\bar{\theta})>0,$$

for all pairs $(\theta, \bar{\theta})$, $0\leqq\theta\leqq 2\pi$, $0\leqq\bar{\theta}\leqq 2\pi$, except when $\theta=\bar{\theta}$.

But this, by Nos. II and IV of our general theorems (see page 120) expresses the following result:

II. *Every extremal-element*† (x, y, ψ) *in the region I is an element of a hyperstrong minimum.*

This result could have been obtained more simply, by observing that since $F_1>0$, for all values of ψ between 0 and 2π, condition (IV′) is satisfied.

It follows further, that the curve does not admit double tangents, so that *no point in the region I can be the corner of a discontinuous solution,* which is evident by theorem D (page 123).

*For the first part of this paper, see the previous number of this journal.

†To say, that (x, y, ψ) is an extremal element is a short way of expressing, that it is possible to draw through the point (x, y) an extremal in a direction, which makes an angle ψ with the radius-vector at (x, y), and consequently an angle θ with the positive X-axis (compare p. 124, equation (9) and Fig. 2).

§ 6. *Case II:* $a<r<\sqrt{3}$.
We have here:

$$\begin{aligned} &F(r,\ \psi)>0,\ \ 0\leqq\psi\leqq 2\pi,\\ &F_1(r,\ \psi)>0,\ \ \text{for } 0\leqq\psi<\alpha,\\ &F_1(r,\ \psi)=0,\ \ \text{for } \psi=\alpha,\\ &F_1(r,\ \psi)<0,\ \ \text{for } \alpha<\psi\leqq\tfrac{1}{2}\pi,\end{aligned}$$

where α is defined by

$$\alpha\equiv\sin^{-1}\left(\frac{a}{r}\right),\quad 0\leqq\alpha\leqq\frac{\pi}{2}. \tag{12}$$

Applying theorems *A* and *B* to these results, we find, that the Indicatrix for a point in this region is a curve, closed around the origin, which changes curvature at $\psi=\alpha$, $\pi-\alpha$, $\pi+\alpha$ and $2\pi-\alpha$, having positive curvature in the intervals $(2\pi-\alpha,\ \alpha)$ and $(\pi-\alpha,\ \pi+\alpha)$, and negative curvature in the intervals $(\alpha,\ \pi-\alpha)$ and $(\pi+\alpha,\ 2\pi-\alpha)$.

The form of equation (11) shows us further, that the curve is symmetric with respect to the *X*-axis and with respect the *Y*-axis. To determine its character more accurately, we denote by $\bar{\psi}$ the angle between tangent and radius vector and obtain:

$$\tan.\bar{\psi}=\frac{\rho}{\rho'}=\frac{F}{F'}=\frac{4(1+r^2\sin^2\psi)-(r^2+1)\sqrt{(1+r^2\sin^2\psi)}}{4r^2\sin\psi\cos\psi} \tag{13}.$$

Since $4(1+r^2\sin^2\psi)-(r^2+1)\sqrt{(1+r^2\sin^2\psi)}=4(r^2+1)\sqrt{(1+r^2\sin^2\psi)}\cdot F$,

and moreover for this case $F>0$, it follows, that $\tan\bar{\psi}\neq 0$.

Further, $\tan\bar{\psi}=\infty$, for $\psi=0$, $\psi=\frac{1}{2}\pi$; *i. e.*, the curve crosses at right angles at $\psi=0$ and at $\psi=\frac{1}{2}\pi$ (see Fig. 5, curve *B*). It is finally of importance, to determine the value of ψ, different from $\frac{1}{2}\pi$, for which the curve has a horizontal tangent. Denoting that value by β and transforming to rectangular coordinates:

$$\left.\begin{aligned}\xi&=\rho\cos\psi\\ \eta&=\rho\sin\psi\end{aligned}\right\}, \tag{14}$$

we obtain:

$$\eta'=\rho'\sin\psi+\rho\cos\psi=\frac{\cos\psi[4-(1+r^2)\sqrt{(1+r^2\sin^2\psi)}]}{4(1+r^2)\sqrt{(1+r^2\sin^2\psi)}F^2}.$$

Consequently,

$$\sin^2\beta=\frac{(r^2+5)\ (3-r^2)}{r^2(1+r^2)^2}. \tag{15}$$

Restricting ourselves in our further discussion of the curve to one quadrant, we find:

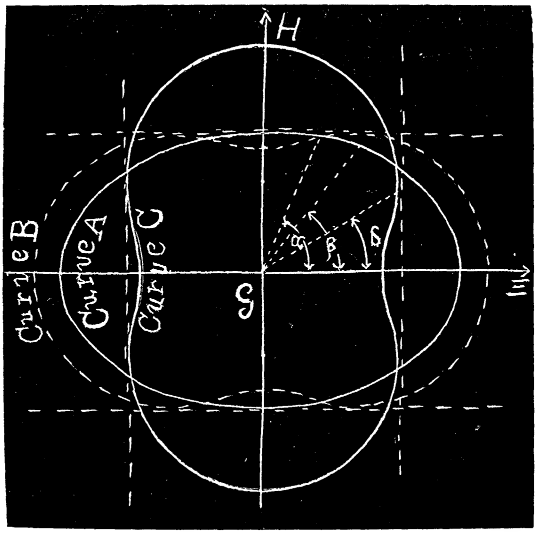

Fig. 5.

For $0 \leqq \psi < \beta$; every tangent lies outside the curve,—it has positive curvature.

For $\psi = \beta$, the curve has a double tangent, touching at β and $\pi - \beta$.

For $\beta < \psi < \alpha$; every tangent intersects the curve,—it has positive curvature.

For $\alpha < \psi < \frac{1}{2}\pi$; every tangent intersects the curve,—it has negative curvature.

By applying rows No. II and IV of our general theorems, we obtain:

$0 \leqq r \sin \psi \leqq r \sin \beta$	$E(x, y; \cos \theta, \sin \theta; \cos \bar{\theta}, \sin \bar{\theta}) < 0$ for $0 \leqq \bar{\theta} \leqq 2\pi$, except $\theta = \bar{\theta}$. $F_1 > 0$.	(x, y, ψ) is an element of strong minimum.
$r \sin \psi = r \sin \beta$	$F_{x'}(x, y, \cos \psi, \sin \psi) = F_{x'}(x, y, \cos(r - \psi), \sin(\pi - \psi))$. $F_{y'}(x, y, \cos \psi, \sin x) = F_{y'}(x, y, \cos(r - \psi), \sin(r - \psi))$.	(x, y, ψ) and $(x, y, r - \psi)$ are elements of discontinuous solutions.
$r \sin \beta < r \sin \psi \leqq a$	$E(x, y; \cos \theta, \sin \theta; \cos \bar{\theta}, \sin \bar{\theta})$ is of changing sign for $0 \leqq \bar{\theta} \leqq 2\pi$. $F_1 > 0$.	(x, y, ψ) is an element of weak minimum.
$a < r \sin \psi < r$	E as above. $F_1 < 0$.	(x, y, ψ) is an element of weak maximum.

Denoting row $r \sin \psi$ by d, and rembering that it represents the distance from the origin to a straight line, we observe that if $r \sin \psi$ has a certain value for one element, it will have the same value along the entire line, to which the element belongs.

We find then, that straight line segments, which lie outside C_a are *weak maxima*, which agrees with a previous result (see I on page 125).

The elements, which belong to lines intersecting C_a, belong to weak minima, if

$$d > \frac{\sqrt{[(r^2+5)(3-r^2)]}}{1+r^2},$$

to strong minima, if

$$d < \frac{\sqrt{[(r^2+5)(3-r^2)]}}{1+r^2}.$$

Solving the equation

$$d = \frac{\sqrt{[(r^2+5)(3-r^2)]}}{1+r^2}$$

for r, we obtain

$$r = \sqrt{\sqrt{\frac{16}{d^2+1}} - 1}, \tag{16}$$

which enables us to determine on every line, intersecting C_a, a point which separates the segments which are strong minima, from those which are weak minima. Constructing this point for all extremals which are parallel to the X-axis, we obtain the curve, denoted by C_r in Figs. 4 and 6. For other extremals, an analogous curve can be obtained by revolving C_r until the line PP' becomes parallel to the given extremal.*

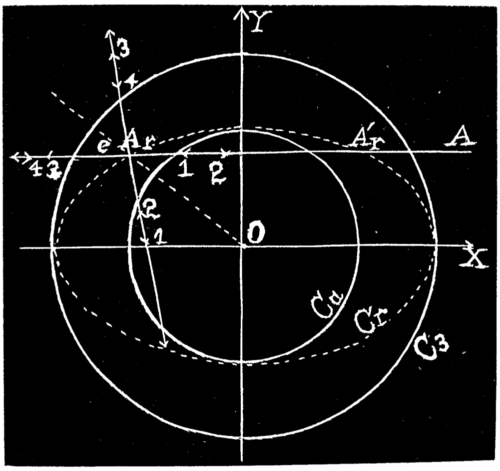

Fig. 6.†

We see further, that since (x, y, φ) is an element of discontinuous solution, every point in this region is a corner of four discontinuous solutions, the branches being in the directions β, $\pi-\beta$, $\pi+\beta$, and $2\pi-\beta$, as indicated in Fig. 6, where 1 and 2, as well as 3 and 4 must be considered as distinct (compare A. Kneser, *Lehrbuch der Variationsrechnung*, p. 9).

From the definition of C_r follows finally, that the intersection of C_r with a line parallel to X-axis (or of an analogous curve with any other line) determine upon that line two points, which are each corners of four discontinuous solutions, of which the given line furnishes one

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*For practical purposes, one would make a cardboard sample of C_r and put it in any required position to determine the points corresponding to A_r and A'_r (Fig. 6).

†ERRATUM.—In the diagram, 4 and 3 on the oblique line through A_r should be interchanged. ED. F.

of the branches. The other branches are obtained by reflecting the line on the radius vector of the point. In Fig. 6, where A is the given line, A_r and A'_r are the points referred to, while the other branches for the former point are formed by BB'.

We have now the following result:

III. *A straight line not intersecting C_a, is a weak maximum within C_3. A straight line intersecting C_a is a strong minimum between C_a and C_r, a weak minimum between C_r and C_3.*

C_r determines upon every line intersecting C_a a corner of a discontinuous solution.

Every point between C_a and C_3 can be a corner of a discontinuous solution.

§ 7. *Case III.* $\sqrt{3}<r<\sqrt{15}$.

In this case we find:

$$F(r,\ \psi)<0, \text{ for } 0\leqq \psi<\gamma,$$
$$F(r,\ \psi)=0, \text{ for } \psi=\gamma,$$
$$F(r,\ \psi)>0, \text{ for } \gamma<\psi\leqq \tfrac{1}{2}\pi,$$

where γ is defined by

$$\frac{\sqrt{(1+r^2\sin^2\gamma)}}{1+r^2}-\tfrac{1}{4}=0,\ 0\leqq\gamma\leqq\tfrac{1}{2}\pi;\ i.\ e.,\ \sin^2\gamma=\frac{(r^2+5)\ (r^2-3)}{16r^2} \tag{17}$$

$F_1(r,\ \psi)$ as in § 6.

Application of theorems A and B shows, that in this case, the Indicatrix has negative curvature in the interval $0\leqq\psi<\gamma$, and has a discontinuity at $\psi=\gamma$. It has further positive curvature from $\psi=\gamma$ to $\psi=\alpha$, from where on, it continues with negative curvature until $\psi=\frac{1}{2}\pi+\alpha$.* The further course of the curve is determined by its symmetry properties.

From equation (13) follows again, that the curve crosses at right angles at $\psi=0$, and at $\psi=\frac{1}{2}\pi$; further $\tan\bar{\psi}=0$, for $\psi=\beta$, *i. e.*, the curve is tangent to a line parallel to the radius vector at its point of discontinuity. To determine the asymptote, we compare the curve at $\psi=\beta$ and the straight line $\psi=\beta$. Denoting the rectilinear co-ordinates of the curve by ξ and η, and those of the straight line by $\bar{\xi}$ and $\bar{\eta}$, we find for a given $\xi=\bar{\xi}$:

$$\eta=\rho\sin\psi=\frac{\sin\psi}{F},\quad \bar{\eta}=\tan\beta.\bar{\xi}=\tan\beta.\xi=\frac{\tan\beta\cos\psi}{F}.$$

*This discussion is made under the assumption that $\gamma<\alpha$; if $\alpha<\gamma$, the curve is of a somewhat different nature. This does not affect the conclusions derived in this consideration however, for which reason the cases are not distinguished. Comparing equations (12) and (17), we find that

$$\alpha \gtreqless \gamma,\ \text{for}\ r \lesseqgtr \sqrt{(4^{\frac{4}{3}}-1)}.$$

Consequently:

$$\eta-\bar{\eta}=\frac{\sin(\psi-\beta)}{\cos\beta\,\mathrm{F}},$$

and $$\underset{\psi=\beta}{\mathrm{L}}(\eta-\bar{\eta})=\frac{16(r^2+1)}{(15-r^2)\sqrt{[(r^2+5)(r^2-3)]}}, \tag{18}$$

which is real and positive for all points between C_3 and C_{15}. It follows, that the asymptote is parallel to the line $\psi=\beta$, and intersects the Y-axis at the point:

$$\eta=\frac{16(r^2+1)}{(15-r^2)\sqrt{[(r^2+5)(r^2-3)]}}.$$

We conclude now immediately, that every tangent intersects the curve, and that no double tangents are possible, from which by theorems C, D and IV, V, and by the reasoning given in § 6 for the connection between extremal elements and extremals, we derive the following results:

IV. *A straight line, not intersecting C_a is a weak maximum between C_3 and C_{15}.*

A straight line, intersecting C_a is a weak minimum between C_3 and C_{15}.

No point in the region between C_3 and C_{15} can be a corner of a discontinuous solution.

§ 8. *Case IV.* $\sqrt{15}<r$.

We have here finally:

$\mathrm{F}(r, \psi)<0,\ 0\leqq\psi\leqq 2\pi.$

$\mathrm{F}_1(r, \psi)$ as in § 6.

We find here, by application of theorems A and B, that the Indicatrix is closed around the origin, has negative curvature for $0\leqq\psi<\alpha$, and positive curvature for $\alpha<\psi\leqq\frac{1}{2}\pi$, again having double symmetry. As in case II, the curve crosses at right angles for $\psi=0$ and $\psi=\frac{1}{2}\pi$, and we want to determine the value of ψ, different from 0 for which it has a vertical tangent. Denoting that value by δ, we find by transforming to rectangular coordinates (14), by a method similar to the one used for the determination of β:

$$\sin^2\delta=\frac{15}{r^2}, \tag{19}$$

which furnishes a real value of δ for any point outside C_{15} (see Fig. 5, curve C).

The case bears very much likeness to the one discussed in § 6, for which reason, we conclude briefly, using theorems C, D, IV, and V:

$0 \leqq r \sin.\psi \leqq a$	(x, y, ψ) is an element of weak minimum.
$a < r \sin.\psi < \sqrt{15}$	(x, y, ψ) is an element of weak maximum.
$r \sin.\psi = \sqrt{15}$	(x, y, ψ) and $(x, y, 2\pi - \psi)$ are elements of discontinuous solutions.
$\sqrt{15} < r \sin.\psi$	(x, y, ψ) is an element of strong maximum.

We notice, that the branches of the discontinuous solutions are determined by the tangents to C_{15}, so that, although every point of the region is a corner of four discontinuous solutions (see Fig. 7), we can not determine on *every* line a corner of a discontinuous solution of which the given line is a branch, as could be done in case II, whereas on some lines we can determine an infinitude of corners, which was impossible in II.

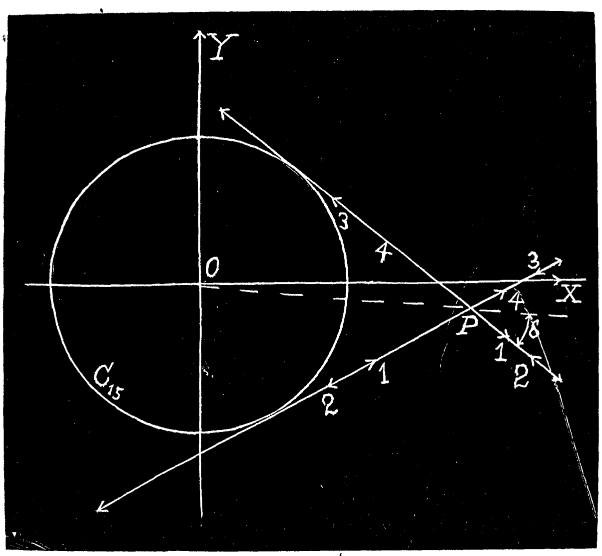

Fig. 7.*

We have here the following result:

V. A straight line intersecting C_a is a weak minimum outside C_{15}. A straight line, not intersecting C_a, but intersecting C_{15}, is a weak maximum outside C_{15}. A straight line, lying entirely outside C_{15}, is a strong maximum.

Every point outside C_{15} is a corner of four discontinuous solutions, which can be determined by drawiag the tangents to C_{15}.

§ 9. *Limiting cases and final results.*

We consider in conclusion:

*Erratum.—δ should be between the dotted line and the lower oblique line through P.

(a) $r=a$. We have then:

$F(r, \psi)>0$, $0 \leqq \psi \leqq 2\pi$.

$F_1(r, \psi)>0$, $0 \leqq \psi \leqq 2\pi$, excepting $\psi=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$.

The Indicatrix is of the same nature as for case I. Consequently, any extremal-element on C_a is an element of a *strong minimum*, while no discontinuous solutions can have a corner on C_a.

(b) $r=\sqrt{3}$.

$F(r, \psi)=0$ for $\psi=0$.

$F(r, \psi)>0$ for $0<\psi \leqq \frac{1}{2}\pi$.

$F_1(r, \psi)$ as in § 6.

The Indicatrix is of the nature described in § 7, and we find from equation (18):

$$\underset{\psi \doteq \beta}{\mathrm{L}} (\eta-\bar{\eta})=\infty,$$

from which follows, that the curve has a parabolic character, so that we conclude: Any extremal-element in C_3 is an element of a *weak minimum*, if $\psi<\alpha$; it is an element of a *weak maximum*, if $\psi>\alpha$.

(c) $r=\sqrt{15}$.

$F(r, \psi)>0$, for $0 \leqq \psi<\frac{1}{2}\pi$.

$F(r, \psi)=0$, for $\psi=\frac{1}{2}\pi$.

$F_1(r, \psi)$ as in § 6.

We find again from equation (19):

$$\underset{\psi=\beta}{\mathrm{L}} (\eta-\bar{\eta})=\infty,$$

from which we derive the same results as in sub (b).

We combine now the results I, II, III, IV, and V, obtained §§ 3, 5, 6, 7, 8 for lines parallel to the X-axis (see Fig. 4):

1. *D is a strong maximum along its entire length and allows four discontinuous solutions at every point.*

2. *C is a weak maximum along its entire length; every point outside of the interval $C'_{15}C_{15}$ is a corner of four discontinuous solutions.*

3. *B is a weak maximum along its entire length; every point on the segment B'_3B_3 is a corner of four discontinuous solutions.*

4. *A is a weak minimum outside the segment* A'_rA_r; it is a strong minimum from A_a up to A_r and from A'_a up to A'_r; it is a hyperstrong minimum on the segment A'_aA_a; all points outside the interval $A'_{15}A_{15}$ and all points on the segments $A'_3A'_a$ and A_3A_a are the corners of four discontinuous solutions; A_r in particular is the corner of the discontinuous solutions, of which one branch lies along A.*

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*The segment AB excludes the points A and B; the interval AB includes the points A and B.

ON THE TRISECTION OF AN ANGLE.

By E. E. WHITE, M. E., Harvard University.

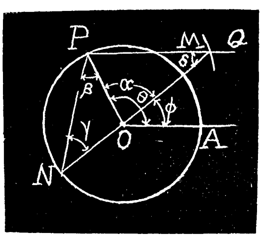
The two following approximate constructions for the trisection of an angle may be interesting. They are, of course, essentially methods of trial and error, since it is only after placing the straight edge in position, passing through the given point, that one can tell whether the position is that desired. The straight-edge must be continually shifted till all the conditions are fulfilled, as nearly as can be told by eye, and mathematically speaking the methods are therefore only approximate.

While it is well known that an angle cannot be trisected by *strict* geometrical construction, nevertheless there are many interesting and practically useful approximate constructions, of which the following are examples. The principle of neither construction is new, though the author devised the first method independently. Four years ago Mr. John J. Quinn made two linkages which involve these constructions, and the linkages in turn depend upon the limaçon. Further, the construction using a graduated scale is an application of the method devised by Archimedes, known as the "Method of Insertions."

CONSTRUCTION I.

Given angle θ, to trisect the angle.

With the vertex, O, as a center, describe a circle of any radius, intersecting the sides of the angle at P and A. Through P draw the line PQ parallel to OA by the usual method. Pass a straight-edge through O, and adjust the position of the straight-edge so that its intersection N with the circle O, and its intersection M with the line PQ shall be equidistant from P (as tested by compasses). Draw the line NOM, which will trisect the angle.



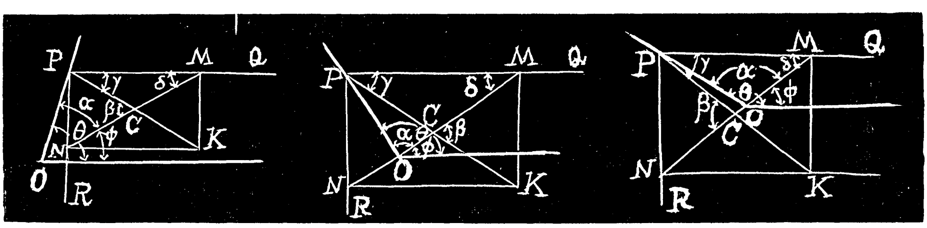
Proof. Draw the line PN.

Then $\theta=\alpha+\phi=\beta+\gamma+\phi=2\gamma+\phi=2\delta+\phi=2\phi+\phi=3\phi$.

CONSTRUCTION II.

Given angle θ, to trisect the angle.

Through any point P on one side of the angle draw PQ parallel and



PR perpendicular to the opposite side of the angle. Mark off a distance equal to $2\overline{OP}$ on a straight-edge passing through O, and adjust the straight-edge so that one mark falls on line PQ at M, and the other on line PR at N. Draw the line NOM, which will trisect the angle.

Proof: Complete the rectangle $PMKN$, and draw the other diagonal PK. Then $\overline{OP}=\frac{1}{2}\overline{MN}=\frac{1}{2}\overline{PK}=\overline{PC}=\overline{CM}$.

Hence, $\theta=\alpha+\phi=\beta+\phi=\gamma+\delta+\phi=2\delta+\phi=2\phi+\phi=3\phi$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

283. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Solve $w+x+y+z=4a$, $w^2+x^2+y^2+z^2=4a^2+4b^2$, $w^3+x^3+y^3+z^3=4a^3+12ab^2$, $w^4+x^4+y^4+z^4=4a^4+4b^4+4c^4+24a^2b^2$.

Solution by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

The following method applies equally well to the corresponding equations with arbitrary constant terms. We are given s_1, s_2, s_3, s_4, where s_n is the sum of the nth powers of w, x, y, z. Hence the latter are, by Newton's identities, the roots of the following quartic:

$$\xi^4-4a\xi^3+(6a^2-2b^2)\xi^2+(4ab^2-4a^3)\xi+a^4+b^4-2a^2b^2-c^4=0.$$

To obtain the reduced quartic, set $\xi=\eta+a$. Then

$$\eta^4-2b^2\eta^2+b^4-c^4=0, \quad (\eta^2-b^2)^2=c^4.$$

Hence, the 24 sets of solutions are given by the arrangements of $a\pm\sqrt{(b^2\pm c^2)}$.

Similarly solved by G. B. M. Zerr and J. Scheffer.

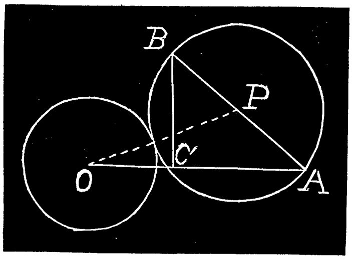
GEOMETRY.

312. Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania, Philadelphia, Pa.

A variable circle passes through a fixed point and is tangent to a given circle. If a diameter of the first circle passes through the fixed point, find the locus of its other extremity.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let O be the center of the given circle, radius$=r$, A the given point, $OA=b$, $OC=x$, $BC=y$. Denoting the radius of the variable circle P by ρ, then



$$(r+\rho)^2=\tfrac{1}{2}y^2+\tfrac{1}{4}(b+x)^2 \text{ and } 4\rho^2=y^2+(b-x)^2.$$

Eliminating ρ, we obtain the equation of the locus of B,

$$y^2=\frac{b^2-r^2}{r^2}(x^2-r^2),$$

a hyperbola, with semi-conjugate axes r and $\sqrt{(b^2-r^2)}$. Consequently, A will be a focus.

Also solved by J. S. Brown.

313. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that an algebraic curve of odd degree which is symmetrical with respect to a center has the center on the curve.

I. Solution by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

By a transformation of coordinates we may assume that the center is at the origin. If the resulting equation of odd degree is $f(x, y)=0$, then must $f(-x, -y)=0$ for all values x, y satisfying $f(x, y)=0$. But the terms of highest degree in $f(x, y)$ change sign when x and y are changed in sign. Hence $f(-x, -y)\equiv -f(x, y)$, so that the terms of even degree vanish. In particular, the constant term is zero, so that the origin lies on the curve.

II. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

If in the equation of any locus we put $-x$, $-y$ for x, y, respectively, we get a locus symmetrical to the given locus with respect to the origin. In case the locus is symmetrical to itself, the latter equation will represent the same locus and differ from the original equation only by a constant multiplier. Let

$$0=u_0+u_1+u_2+u_3+\ldots+u_r+\ldots$$

be the equation of a locus symmetrical with respect to the origin, u_n denoting the terms of degree n, and r being an *odd* integer. Another equation to this locus, obtained by putting $-x$, $-y$ for x, y, respectively, and changing the sign of each term, is

$$0=-u_0+u_1-u_2+u_3-\ldots+u_r-\ldots$$

Since each equation contains the term u_r, the constant multiplier is unity.

Therefore

$$0=u_0=u_2=u_4=\dots$$

That is, *if a locus symmetrical with respect to the origin contains one term of odd degree, it contains no absolute term and no terms of even degree.* A further expansion of the problem leads to the theorem that *if a locus symmetrical with respect to the origin contains one term of even degree, all the terms are of even degree.*

CALCULUS.

239. Proposed by L. H. MacDONALD, A. M., Ph. D., Sometime Tutor in the University of Cambridge, Jersey City, N. J.

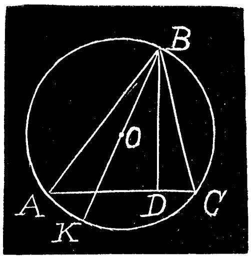
Of all triangles inscribed in a circle, find that which has the greatest perimeter.

Solution by C. N. SCHMALL, 89 Columbia Street, New York City, and REV. J. H. MEYER, S. J., Augusta, Ga.

Let ABC be the required triangle; O, the center of the given circle; BK, $=2r$, the diameter of the circle; $\angle BAC=\phi$, and $\angle BCA=\psi$. Then the perimeter, p, $=AB+BC+AC=$maximum... (1).

But $AB.BC=2r.BD$, whence $BC=2r\dfrac{BD}{AB}=2r\sin\phi$.

Also $\dfrac{AB}{BC}=\dfrac{\sin\psi}{\sin\phi}$, whence $AB=BC\dfrac{\sin\psi}{\sin\phi}$, which, from the previous equation, $=2r\sin\psi$. We also have, from the law of signs,



$$AC=\frac{\sin(\phi+\psi)}{\sin\phi}BC=2r\sin(\phi+\psi).$$

Substituting in (1), we have

$$p=2r[\sin\phi+\sin\psi+\sin(\phi+\psi)]\dots(2).$$

Hence, for a maximum or minimum,

$$\frac{\partial p}{\partial\phi}=\cos\phi+\cos(\phi+\psi)=0,\ \text{and}\ \frac{\partial p}{\partial\psi}=\cos\psi+\cos(\phi+\psi)=0.$$

Hence, $\cos\phi=\cos\psi$ and $\phi=\psi$. Since $\cos\phi+\cos(\phi+\psi)=0$, we have, by substitution, $\cos\phi+\cos2\phi=0$, or $\cos\phi+2\cos^2\phi-1=0$, whence, $\cos\phi=\frac{1}{2}$, and therefore $\phi=60^\circ=\psi$.

Hence, the triangle is equilateral.

It is easy to show that $\dfrac{\partial^2p}{\partial\phi^2}.\dfrac{\partial^2p}{\partial\psi}>\dfrac{\partial^2p}{\partial\phi\,\partial\psi}$, and that, therefore, the triangle is a maximum.

Also solved by A. F. Carpenter, G. B. M. Zerr, A. H. Holmes, J. Scheffer, and G. W. Greenwood.

Professors Zerr, Greenwood, and Scheffer, and Mr. Holmes denoted the angles at the center subtended by the sides, by 2θ, 2ϕ, and 2ψ, and showed that these angles are equal to 120° each. Professor Carpenter showed that the inscribed triangle with one side constant and of maximum perimeter is isosceles, and then showed that of all isosceles triangles inscribed in a circle, the equilateral has the maximum perimeter.

240. Proposed by L. MORDELL, Philadelphia, Pa.

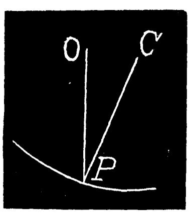
Show that the osculating conic of the catenary $y=c\cosh\frac{x}{c}$ at the point for which $y=\frac{c\sqrt{10}}{2}$ is a parabola.

I. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let O be the center of curvature of the point considered, and C the center of the conic of closed contact, then (*vide* Joseph Edwards' *Differential Calculus*, where the problem is proposed as an exercise):

$$\frac{\cos\phi}{R}=\frac{1}{\rho}-\frac{\partial\phi}{\partial s},\quad\text{and}\quad \phi=\tan^{-1}\tfrac{1}{3}.\frac{\partial\rho}{\partial s},$$

where $\phi=\angle OPC$, $OP=\rho=$radius of curvature, s an arc of the given curve, and $PC=R$. From



$$y=\frac{c}{2}(e^{x/c}+e^{-(x/c)}),\ \text{we find}\ \frac{\partial y}{\partial x}=\tfrac{1}{2}(e^{x/c}-e^{-(x/c)}),$$

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{2c}(e^{x/c}+e^{-(x/c)},\quad \frac{\partial s}{\partial x}=\tfrac{1}{2}(e^{x/c}+e^{-(x/c)}).$$

$$\therefore \rho=\left[\left(\frac{\partial s}{\partial x}\right)^3\Big/\left(\frac{\partial^2 y}{\partial x^2}\right)\right]=\frac{c}{4}(e^{x/c}+e^{-(x/c)})^2.\quad \therefore\frac{\partial\rho}{\partial s}=\frac{\partial\rho}{\partial x}.\frac{\partial x}{\partial s}=e^{x/c}-e^{-(x/c)}$$

therefore, $\phi=\tan^{-1}\frac{e^{x/c}-e^{-(x/c)}}{3}$, or, $e^{x/c}-e^{-(x/c)}=3\tan\phi$.

$$\therefore\frac{\partial\phi}{\partial x}=\frac{(e^{x/c}+e^{-(x/c)})\cos^2\phi}{3c}.\quad\therefore\frac{\partial\phi}{\partial s}=\frac{\partial\phi}{\partial x}.\frac{\partial x}{\partial s}=\frac{2}{3c}\cos^2\phi.$$

For the value $y=\frac{c}{2}\sqrt{10}$, we find $\tan\phi=\frac{1}{3}\sqrt{6}$, $\cos^2\phi=\frac{3}{5}$, $\rho=\frac{5c}{2}$, $\frac{\partial\phi}{\partial s}=\frac{2}{5c}$; substituting in $\frac{\cos\phi}{R}=\frac{1}{\rho}-\frac{\partial\phi}{\partial s}$, we find $\frac{\sqrt{\frac{3}{5}}}{R}=\frac{2}{\rho c}-\frac{2}{\rho c}=0$; $\therefore R=\infty$,

and since the conic, the center of which is at an infinite distance, is a parabola, the assertion is proved.

II. **Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.**

In the catenary, $y=c\sec\psi$, we have $s=c\tan\psi$ and $\rho=c\sec^2\psi$.

If a conic whose center is C osculates a curve at a point P, the center of curvature for the curve at that point being O, we have

$$\tan\phi=\tfrac{1}{3}.\frac{d\rho}{ds},\text{ and }\frac{\cos\phi}{PC}=\frac{1}{\rho}-\frac{d\phi}{ds},\text{ where }\phi=\angle OPC.$$

[Edwards' *Differential Calculus*, p. 356, 3d edition.]

$$\rho=c(1+\tan^2\psi)=c+\frac{s^2}{c},\quad \frac{d\rho}{ds}=2\tan\psi.$$

$$\therefore \tan\phi=\tfrac{2}{3}\tan\psi;\ (1+\tan^2\phi)d\phi=\tfrac{2}{3}(1+\tan^2\psi)d\psi.$$

$$\therefore \frac{\cos\phi}{PC}=\frac{1}{\rho}-\frac{d\phi}{d\psi}\cdot\frac{1}{\rho}=\frac{1}{\rho}\left(1-\frac{6(1+\tan^2\psi)}{9+4\tan^2\psi}\right)=\frac{3-2\tan^2\psi}{\rho(9+4\tan^2\psi)}=0,$$

when $y=c\sec\psi=\dfrac{c\sqrt{10}}{2}$.

Hence (since $\cos\phi\neq0$), PC is infinite, and the conic is a parabola.

This question was set at Oxford in 1889.

Also solved by G. B. M. Zerr.

MECHANICS.

200. **Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.**

An elastic string whose weight is W is laid over the top of an inclined plane so as to remain at rest. Determine how much the string will be elongated, knowing, M=modulus of elasticity, L=normal length of string, and ϕ=inclination of the plane.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let M=mass of unit length of the unstretched string. The equation of equilibrium is $dT+Mg\sin\varphi ds=0$. But $MgL=W$.

$$\therefore T+\frac{W\sin\varphi s}{L}=\text{constant. Now when } s=0,\ T=W\sin\varphi.$$

$$\therefore T=W\sin\varphi(1-\frac{s}{L}).\quad\text{Also } dl=(1+\frac{T}{M})ds=[1+W\sin\varphi(\frac{1}{M}-\frac{s}{LM})]ds.$$

$$l=\int_0^L[1+W\sin\varphi(\frac{1}{M}-\frac{s}{LM})]ds=L[1+W\sin\varphi(\frac{1}{M}-\frac{1}{2M})]$$

$$=L[1+\frac{W\sin\varphi}{2M}].\quad \therefore \frac{WL\sin\varphi}{2M}\text{ is the elongation.}$$

Also solved by the Proposer.

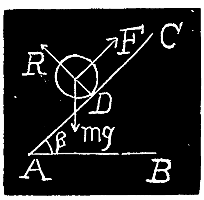
201. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

ABC is an inclined plane, perfectly rough, length $AC=l$. The time for a sphere to roll down when AB is base is to the time for a cylinder to roll down when BC is base as m is to n. Find AB and BC.

Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let O be the center of any rolling body of mass m; then the three forces that will act on it are its weight mg vertically downward, the resistance R on the inclined plane AC, and the friction F acting up the plane.

Denoting CD by s, we have therefore $m(\partial^2 s/dt^2)=mg\sin\beta-F$; and if we denote by θ the angular velocity, reducing the mass to the center O, we have $m\rho^2(\partial^2\theta/\partial t^2)=Fr$, r being equal to DO, and ρ radius of gyration. Since there is no sliding, the plane being perfectly rough, we have $s=r\theta$. Eliminating F, we have $\partial^2 s/dt^2=[r^2/(r^2+\rho^2)]g\sin\beta$, and integrating



$$s=\frac{r^2}{r^2+\rho^2}g\sin\beta\, t^2.$$

For the sphere $\rho^2=\frac{2}{5}r^2$, and for the cylinder $\rho^2=\frac{1}{2}r^2$; therefore, by the condition of the problem, for the sphere

$$l=\frac{r^2}{r^2+\frac{2}{5}r^2}g\,.\frac{BC}{l}t^2,\text{ whence } t^2=\frac{14l^2}{5g.\overline{BC}},$$

and for the cylinder, $l=\dfrac{r^2}{r^2+\frac{1}{2}r^2}g\,.\dfrac{AB}{l}t^2$, whence $t^2=\dfrac{3l^2}{g.\overline{AB}}$; $\therefore \dfrac{14}{5\overline{BC}}:\dfrac{3}{\overline{AB}}=m^2$

$:n^2$, and combining this with $\overline{AB}^2+\overline{CB}^2=l^2$, we get

$$AB=\frac{14n^2l}{\sqrt{(225m^4+196n^4)}},\quad CB=\frac{15m^2l}{\sqrt{(225m^4+196n^4)}}.$$

Also solved by J. Edward Sanders, and the Proposer.

AVERAGE AND PROBABILITY.

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185. Proposed by R. D. CARMICHAEL, Anniston, Ala.

If a line l is divided into n parts by $n-1$ points taken at random on it, what is the mean value of the pth power of one of the parts taken at random?

Solution by HENRY HEATON, Belfield, N. D.

By making the line the circumference of a circle it will be seen that each part has exactly the same chance, so that the average of one part is the same as that of any other. Hence it will be sufficient to find the average length of any one part.

Let $x_1, x_2, \ldots, x_{n-1}$ be the distance of the different points from one extremity of the line. Then the average length of $(l-x_{n-1})^p$ is

$$A=\frac{\int_0^l\int_0^{x_{n-1}}\cdots\int_0^{x_2}(l-x_{n-1})^p dx_{n-1}dx_{n-2}\cdots dx_1}{\int_0^l\int_0^{x_{n-1}}\cdots\int_0^{x_2}dx_{n-1}dx_{n-2}\cdots dx_1}$$

$$=\frac{\int_0^l(l-x_{n-1})^p x_{n-1}^{n-2}\,dx_{n-1}}{\int_0^l x_{n-1}^{n-2}dx_{n-1}}=(n-1)l^pB[(p+1),\ (n-1)]$$

$$=\frac{(n-1)l^p\,[\Gamma(p+1),\ \Gamma(n-1)]}{\Gamma(p+n)}=\frac{l^p\,[\Gamma(p+1),\ \Gamma(n)]}{\Gamma(p+n)}.$$

or, the average length of $x_1{}^p$ is

$$A=\frac{\int_0^l\int_0^{x_{n-1}}\cdots\int_0^{x_2}dx_{n-1}\,dx_{n-2}\cdots x_1{}^p dx_1}{\int_0^l\int_0^{x_{n-1}}\cdots\int_0^{x_2}dx_{n-1}\,dx_{n-2}\cdots dx_1}$$

$$=\frac{l^p(n-1)!\,p!}{(p+n-1)!}=\frac{l^p[\Gamma(n),\ \Gamma(p+1)]}{\Gamma(p+n)}.$$

Also solved by G. B. M. Zerr.

186. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

An urn contains $n=100$ balls; $a=25$ balls are stamped, at random, with the letter A; $b=30$ balls are stamped, at random, with the letter B; $c=40$ balls are stamped, at random, with the letter C; $d=50$ balls are stamped, at random, with the letter D. One ball is drawn at random; find the chance it has on it no letter, the letter A, or B, or C, or D, or the letters AB, AC, AD, BC, BD, CD, ABC, ABD, ACD, BCD, or $ABCD$.

Solution by J. E. SANDERS, Reinersville, O., and the PROPOSER.

$(a/n)=\frac{25}{100}=\frac{1}{4}$, $(1-a/n)=\frac{3}{4}$; $(b/n)=\frac{30}{100}=\frac{3}{10}$, $(1-b/n)=\frac{7}{10}$; $(c/n)=\frac{40}{100}=\frac{2}{5}$, $(1-c/n)=\frac{3}{5}$, $(d/n)=(1-d/n)=\frac{50}{100}=\frac{1}{2}$.

$\therefore$ The required chances are in order:

$(1-a/n)(1-b/n)(1-c/n)(1-d/n)=\frac{3}{4}.\frac{7}{10}.\frac{3}{5}.\frac{1}{2}=\frac{63}{400}.$
$(a/n)(1-b/n)(1-c/n)(1-d/n)=\frac{1}{4}.\frac{7}{10}.\frac{3}{5}.\frac{1}{2}=\frac{21}{400}.$
$(1-a/n)(b/n)(1-c/n)(1-d/n)=\frac{3}{4}.\frac{3}{10}.\frac{3}{5}.\frac{1}{2}=\frac{27}{400}.$
$(1-a/n)(1-b/n)(c/n)(1-d/n)=\frac{3}{4}.\frac{7}{10}.\frac{2}{5}.\frac{1}{2}=\frac{42}{400}.$
$(1-a/n)(1-b/n)(1-c/n)(d/n)=\frac{3}{4}.\frac{7}{10}.\frac{3}{5}.\frac{1}{2}=\frac{63}{400}.$
$(a/n)(b/n)(1-c/n)(1-d/n)=\frac{1}{4}.\frac{3}{10}.\frac{3}{5}.\frac{1}{2}=\frac{4}{400}.$
$(a/n)(1-b/n)(c/n)(1-d/n)=\frac{1}{4}.\frac{7}{10}.\frac{2}{5}.\frac{1}{2}=\frac{14}{400}.$
$(a/n)(1-b/n)(1-c/n)(d/n)=\frac{1}{4}.\frac{7}{10}.\frac{3}{5}.\frac{1}{2}=\frac{21}{400}.$
$(1-a/n)(b/n)(c/n)(1-d/n)=\frac{3}{4}.\frac{3}{10}.\frac{2}{5}.\frac{1}{2}=\frac{18}{400}.$
$(1-a/n)(b/n)(1-c/n)(d/n)=\frac{3}{4}.\frac{3}{10}.\frac{3}{5}.\frac{1}{2}=\frac{27}{400}.$
$(1-a/n)(1-b/n)(c/n)(d/n)=\frac{3}{4}.\frac{7}{10}.\frac{2}{5}.\frac{1}{2}=\frac{42}{400}.$
$(a/n)(b/n)(c/n)(1-d/n)=\frac{1}{4}.\frac{3}{10}.\frac{2}{5}.\frac{1}{2}=\frac{6}{400}.$
$(a/n)(b/n)(1-c/n)(d/n)=\frac{1}{4}.\frac{3}{10}.\frac{3}{5}.\frac{1}{2}=\frac{9}{400}.$
$(a/n)(1-b/n)(c/n)(d/n)=\frac{1}{4}.\frac{7}{10}.\frac{2}{5}.\frac{1}{2}=\frac{14}{400}.$
$(1-a/n)(b/n)(c/n)(d/n)=\frac{3}{4}.\frac{3}{10}.\frac{2}{5}.\frac{1}{2}=\frac{18}{400}.$
$(a/n)(b/n)(c/n)(d/n)=\frac{1}{4}.\frac{3}{10}.\frac{2}{5}.\frac{1}{2}=\frac{6}{400}.$

187. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

Solution by the PROPOSER.

Let x and y be the coordinates of one of the points, and θ=the angle one of the lines makes with the side taken as the Y-axis. Then the length of the line from the point to the side taken as the X-axis is $y\sec\theta$.

It is only necessary to find the average length of this line, for the average length of lines drawn to the other sides is evidently the same.

When the line passes through the origin, $y\tan\theta=x$. Hence the limits of θ are 0 and $\tan^{-1}(x/y)$. Hence, the required average is,

$$A=\int_0^a\int_0^a\int_0^{\tan^{-1}(x/y)} y\sec\theta\,dx\,dy\,d\theta\div\int_0^a\int_0^a\int_0^{\tan^{-1}(x/y)} dx\,dy\,d\theta$$

$$=\int_0^a\int_0^a \log\left(\frac{x}{y}+\frac{\sqrt{(x^2+y^2)}}{y}\right)dxy\,dy\div\int_0^a\int_0^a \tan^{-1}\left(\frac{x}{y}\right)dx\,dy$$

$$=\int_0^a\left[\frac{a^2}{2}\log\left(\frac{x+\sqrt{(a^2+x^2)}}{a}\right)+\frac{x}{2}\sqrt{(a^2+x^2)}-\frac{x^2}{2}\right]dx$$

$$\div\int_0^a\left[a\tan^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\log\left(\frac{a^2+x^2}{x^2}\right)\right]dx=\frac{2a}{\pi}[\log(1+\sqrt{2})-\tfrac{1}{3}(\sqrt{2}-1)]$$

$$=.4095a.$$

Also solved by G. B. M. Zerr, whose solution will appear in the next issue.

PROBLEMS FOR SOLUTION.

ALGEBRA.

287. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

For what fraction of a year will there be the greatest difference between the interest as computed by the ordinary commercial rule and that computed by the rule of compound interest?

288. Proposed by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.

Evaluate the determinant which arises in finding the inverse of the transformation, with binomial coefficients,

$$T: \quad \xi_i = \sum_{j=i}^{g-1} \binom{j}{i} x_j \quad (i=0,\ 1,\ \ldots,\ g-1).$$

GEOMETRY.

320. Proposed by S. F. NORRIS, Baltimore City College.

Lines are drawn from a fixed point P_1, meeting a fixed circle in P_2. On P_1P_2 a point P is taken so that $P_1P \times P_1P_2 = k^2$. Find the locus of P. Solve by analytic methods, using rectangular coordinates, and putting the result in the form,

$$(x_1^2+y_1^2-r^2)[(x-x_1)^2+(y-y_1)^2]+2k^2(x_1x+y_1y-x_1^2-y_1^2)+k^4=0.$$

321. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Prove by plane geometry the following interesting theorem:

If from a point in the plane of a triangle perpendiculars are demitted upon the three sides of the triangle, and if the area of the triangle formed by connecting the feet of these perpendiculars is denoted by $\triangle'$, *the distance of the assumed point from the center of the circle circumscribed about the original triangle by* R', *the radius of the circumscribed circle by* R, *and the area of the pedal triangle by* $\triangle$, *then will* $\triangle'/\triangle = \pm[(R^2-R'^2)/R^2]$.

CALCULUS.

242. Proposed by J. H. MEYER, S. J., Augusta, Ga.

A given sphere is to be formed into a solid composed of two equal cones on opposite sides of a common base, in such a manner that its surface may be the least possible. Find the dimensions of the solid, and compare its surface with that of the sphere.

243. Proposed by R. D. CARMICHAEL, Anniston, Ala.

The usual method for the solution of a differential equation in the form (see Cohen, *Differential Equations*, p. 22)

$$x^r y^s (my\,dx+nx\,dy) + x^\rho y^\sigma (\mu y\,dx+\nu x\,dy)=0$$

fails when (1) $n=am$, (2) $\nu=a\mu$, (3) $s-\sigma \neq a(r-\rho)$. Find the solution when the relations (1) and (2) hold. (Note that the solution desired does not depend on (3).)

244. Proposed by G. B. M. ZERR, Ph. D., Professor of Mathematics in Central Manual Training School, Philadelphia, Pa.

Fine the volume common to the solids bounded by the surfaces

$$x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3} \text{ and } x^{1/3}z^{2/3}=(a^{1/3}-x^{1/3})(x^{2/3}+y^{2/3}).$$

MECHANICS.

205. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given two points A and B not in the same horizontal nor in the same vertical line; to find the path from A to B along which a particle will slide from rest under the force of gravity alone so that the average velocity along the curve shall be a maximum.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A rigid square $ABDC$ made by smooth wires is fixed with A vertically above D. Two small equal spherical elastic beads slide down BD, CD, starting simultaneously from B and C. Find the ratio of their velocities of approach and separation at D, and how far they will separate after impact.

NOTES AND NEWS.

The eminent mathematician, Yoshio Mikami, is translating Dr. Halsted's *Rational Geometry* for publication in Japan. F.

Dr. G. B. M. Zerr has been appointed Professor of Mathematics in the Central Manual Training School of Philadelphia, at a salary of $2000 per year. F.

Professor G. W. Greenwood has resigned his position as Professor of Mathematics in Roanoke College and gone into business in Dunbar, Pa. He will, however, continue his contributions to the MONTHLY. F.

Miss Hazel Anderson, who has just received the Master's degree at the University of Chicago, will be Instructor in Mathematics at the Manual Training High School, Indianapolis, Ind. S.

Mr. G. R. Clements, who took the Master's degree at the University of Chicago in August, has been appointed to an Instructorship in Mathematics at Williams College, Williamstown, Mass. S.

Miss Mary E. Wells, who received the Master's degree at the University of Chicago in June, returns to an Instructorship in Mathematics at Mt. Holyoke College, where she graduated in 1906. S.

Dr. R. L. Boerger, formerly instructor at the University of Missouri, has been appointed to an Instructorship in Mathematics at the University of Illinois. Mr. Boerger received the Doctor's degree at the University of Chicago in August, 1907. S.

Mr. John W. Mitchell has been appointed Instructor in Matematics at the Agricultural and Mechanical College of Texas. S.

Dr. Louis Ingold returns to the University of Missouri as Instructor in Mathematics, after a two years' leave of absence, during which time he has been Fellow in Mathematics at the University of Chicago, where he has just received the Doctor's degree. S.

Dr. W. H. Bussey, who for two years has been instructor in Barnard College, Columbia University, has been elected to an Assistant Professorship in Mathematics at the University of Minnesota. Mr. Bussey took his Doctorate at the University of Chicago in 1906. S.

Dr. N. J. Lennes has been elected to an Instructorship in Mathematics at the Massachusetts Institute of Technology. Mr. Lennes has been for some years teaching mathematics in the Chicago high schools, and has just taken his Doctorate at the University of Chicago. S.

Dr. F. W. Owens has been made Instructor in Mathematics at Cornell University. For two years Mr. Owens has been teaching mathematics at the Academy of North Western University at Evanston, Ill. He took his Doctorate at the University of Chicago on August 30, 1907. S.

Dr. N. R. Wilson, who received his degree at the University of Chicago at the August Convocation, returns as Associate Professor of Mathematics at the University of Manitoba, Winnipeg, Manitoba. He has been there several years and was recently promoted to Associate Professor. S.

E. J. Wilczynski, Ph. D. (Berlin), Associate Professor of Mathematics in the University of California, has been appointed to a similar position in the University of Illinois. Professor Wilczynski is one of the best mathematicians of America. He has been both research assistant and research associate of the Carnegie Institute of Washington, and the 1904 Year Book contains the following about his work: "The general character of these investigations places them at the beginning of a new kind of geometry, a projective geometry which does not confine itself to the consideration of the algebraic cases, as has hitherto been the case, but which proves theorems of a more general nature by the use of differential equations, resembling in that respect the general theory of surfaces." Professor Wilczynski is chairman of the San Francisco Section of the American Mathematical Society, and is the author of *Projective Differential Geometry of Curves and Ruled Surfaces*, recently published by B. G. Teubner of Germany. M.

ERRATA.

In solution of problem 197, Mechanics, page 107, the following corrections should be noted:

Line 7, for v_1 read v_2; line 8, for v_2 read v_1; line 9, for v_1^2/r read v_2^2/R.

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NOTES ON THE GEOMETRICAL REPRESENTATION OF THE ROOTS OF EQUATIONS.

By DR. O. E. GLENN, The University of Pennsylvania.

The importance, from the historical standpoint, of the method of representing the roots of equations by means of intersecting conics, suggests that this species of graphical representation might deserve a more prominent place in teaching. Constructions of this kind were known to the Greek geometers, an account of whose discoveries may be found in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*, Chap. 11, pp. 240-243. An account of the work in the same direction by Descartes, Van Schooten and others is given in M. Cantor's *Geschichte der Mathematik*, Vol. 2, Chap. LXXVII, pp. 736-737, and a short reference to the methods in Klein's **Vortraege über Ausgewahlte Fragen der Elementargeometrie.*

In this paper constructions for the cubic, biquadratic, and quintic are obtained in what is believed to be their most practicable form. These results are then employed to derive elementary analytical properties of the equations directly from the geometrical properties of the constructions.

§ 1. The Cubic.

The roots of the general cubic

$$x^3+px+q=0 \tag{1}$$

may be obtained by eliminating z from the simultaneous pair

$$x^2+qz+p=0,$$
$$x+qz^2+pz=0,$$

x, z being independent variables. Also the same trio of x values will be obtained from this pair altered by the transformation

$$z=\frac{-y}{q}-\frac{p}{q}.$$

*See Translation by Beman and Smith.

Hence the roots of the cubic (1) are the abscissas of the intersections (apart from the origin) of the parabola and circle

$$(2)\qquad \begin{aligned} P &: y-x^2=0,\\ S &: x^2+y^2+qx+(p-1)y=0. \end{aligned}$$

The parabola P is the same for all cubics, and will be called the primary curve. The circle S always passes through the origin but is different for different cubics. Let us call it the secondary curve. Since the sum of the three abscissas in question is always zero, to each circle through the origin will correspond a cubic equation in form similar to (1) and inversely, so that the totality (∞^2) of all cubics is represented in the plane uniquely by the totality of all circles (∞^2) which pass through the origin O. The secondary S has its center at the rational point

$$\left(-\frac{q}{2}, \frac{1-p}{2}\right)$$

and is always real. Hence, *to construct the roots of the cubic, take* $\left(-\frac{q}{2}, \frac{1-p}{2}\right)$ *as a center* C, *and with* $\overline{CO}$ *as a radius describe a circle* S. *The perpendiculars from the intersections of this circle and* P, *upon the axis of* P, *are the roots of the cubic*

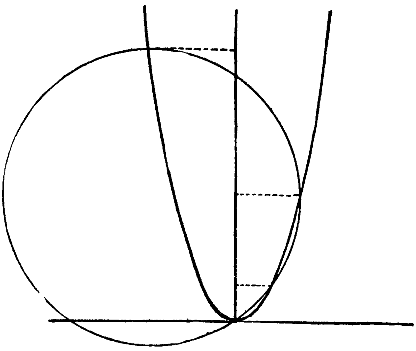
$$x^3+px+q=0.$$

Example: Construct the roots of the equation

$$x^3-7x+6=0.$$

Here we have

$$P : y-x^2=0;\ S : x^2+y^2+6x-8y=0.$$



I. Scale, $\frac{3}{16}$ in.=unity.

From the figure [1], $r_1=-3; r_2=2; r_3=1$.

Equal, Real, and Imaginary Roots.

If there exists a relation connecting p, q

$$(3)\qquad f(p, q)=0$$

the pencil (∞') of circles S will have an envelope. When S is tangent to P, two roots of the cubic are equal. Hence if the envelope corresponding to the given relation (3) contains P as a factor, $f(p, q)$ is the condition for equal roots.

Now f may be determined to fulfill these conditions. For if we consider as known the theorem that the discriminant of an algebraical equation of degree n is an homogeneous function of its n coefficients, of degree $2(n-1)$, we may write the discriminant for $n=3$ in the following form, with undetermined coefficients α:

$$\Delta : \alpha_1+\frac{\alpha_2}{3}p+\alpha_3 q+\frac{\alpha_4}{27}p^3+\frac{\alpha_5}{4}p^2+\frac{\alpha_6}{4}p^2 q+\alpha_7 q^3+\alpha_8 q^2+\frac{\alpha_9}{3}q^2 p+\alpha_{10} pq.$$

The elimination of p, q, $\partial q/\partial p$ between the four equations

$$S=0,\ \Delta=0,\ \frac{\partial S}{\partial p}=0,\ \frac{\partial \Delta}{\partial p}=0,$$

and equating to zero (identically) the result of substituting $y=x^2$ in the result gives

$$\alpha_1=\alpha_2=\alpha_3=\alpha_5=\alpha_6=\alpha_7=\alpha_9=\alpha_{10}=0,\ \alpha_4=1,\ \alpha_8=\tfrac{1}{4}.$$

Hence the condition for equal roots is

$$\frac{q^2}{4}+\frac{p^3}{27}=0.$$

Reality of Roots in Cardan's Irreducible Case.

Let
$$\frac{q^2}{4}+\frac{p^3}{27}=p^2\delta,\ (\delta\doteq 0).$$

Then
$$q=\pm\frac{2}{\sqrt{27}}[(-p)^{\frac{3}{2}}+\tfrac{27}{2}(-p)^{\frac{1}{2}}\,\delta]$$

and
$$S : x^2+y^2\pm\frac{2}{\sqrt{27}}[(-p)^{\frac{3}{2}}+\tfrac{27}{2}(-p)^{\frac{1}{2}}\,\delta]x+(p-1)y=0,$$

$$\frac{\partial S}{\partial p} : \pm\frac{2}{\sqrt{27}}[(-\tfrac{3}{2})(-p)^{\frac{1}{2}}-\tfrac{27}{4}(-p)^{-\frac{1}{2}}\delta]x+y=0,$$

and the envelope is

$$x^4+x^2y^2-y^3-x^2y+9x^2y\,\delta=0.$$

That is

$$y=x^2+\frac{9x^2y}{x^2+y^2}\delta.$$

Hence the ordinate of this envelope is greater than the corresponding ordinate of the primary when δ is positive, and less when δ is negative. Thus the envelope of S is enclosed entirely within the primary, or is entirely enclosed by the primary according as δ is positive or negative. Hence when δ is positive,

$$\frac{q^2}{4}+\frac{p^3}{27}>0,$$

two roots of (1) are imaginary. But when δ is negative,

$$\frac{q^2}{4}+\frac{p^3}{27}<0 \text{ (Cardan's irreducible case),}$$

S intersects P in three real points, so all the roots are real.

§ 2. The Quartic.

The roots of the general quartic

$$x^4+px^2+qx+r=0 \tag{4}$$

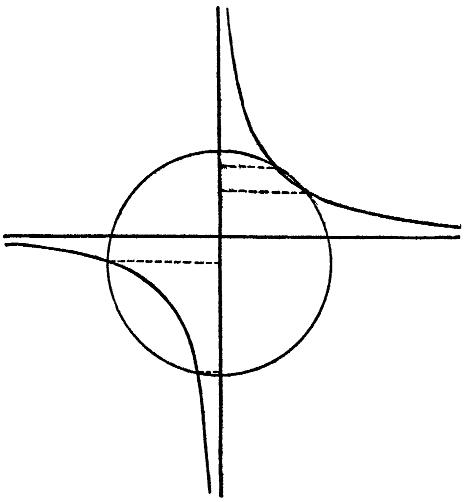
are the abscissas of the intersections of

$$P : xy-1=0,$$
$$S : x^2+ry^2+qy+p=0.$$

The primary (the same for all quartics) is the equilateral hyperbola P. The secondary is a three parameter central conic. Another convenient form is

$$P' : xy-\sqrt{(\epsilon r)}=0,$$
$$S' : x^2+\epsilon y^2+\frac{q}{\sqrt{\epsilon r}}y+p=0,$$

where $\epsilon=1$, -1 as r is positive or negative. If r is positive S' is a circle with center at $\left(0, \frac{-q}{2\sqrt{r}}\right)$. The center of the secondary conic is always on the y axis. The geometrical construction of the roots is now obvious.



II. Scale, $\frac{1}{7}$ in.=unity.

Example: Construct the roots of the quartic

$$x^4-15x^2+10x+24=0.$$

We have

$$P' : xy-2\sqrt{6}=0;$$
$$S' : x^2+y^2+\tfrac{5}{6}\sqrt{6}\,y-15=0.$$

Center $(0, -\frac{5}{12}\sqrt{6})$; radius$=\frac{1}{2}\sqrt{\frac{385}{6}}$. From the figure [II],

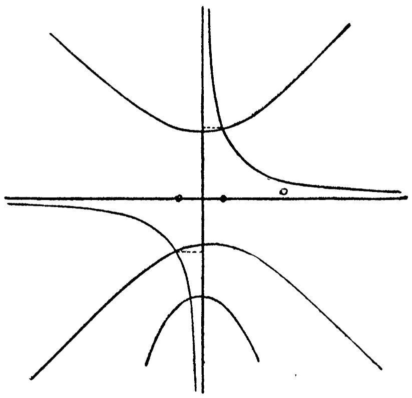
$$r_1=-1,\ r_2=-4,\ r_3=2,\ r_4=3.$$

Harmonic and Equi-Anharmonic Ranges of Roots.

Let us find the envelope of S corresponding to the relation

$$g_2 : \tfrac{1}{12}(p^2+12r)=0,$$

regarding q as invariable. We have from this equation



$x^4+6x^2+x-3=0$

III. Scale, $\frac{1}{2}$ in.=unity.

$$S : x^2 -\frac{p^2}{12}y^2+qy+p=0,$$

$$\frac{\partial S}{\partial p} : -\tfrac{1}{6}y^2p+1=0.$$

Hence $p=\frac{6}{y^2}$, and the envelope is

$$(5) \qquad x^2y^2+qy^3+3=0.$$

We may state the result as follows: *When the secondary conic is tangent to the quartic* (5) *the roots of the biquadratic* (4) *form an equi-anharmonic range.* (See example, Fig. III.)

Next consider p as invariable and let

$$g_3 : \tfrac{1}{6}(pr-\tfrac{3}{8}q^2-\frac{p^3}{36})=0.$$

Then

$$r=\tfrac{3}{8}\frac{q^2}{p}+\frac{p^2}{36},$$

and

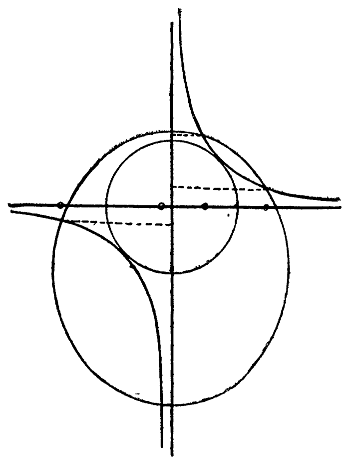
$$S : x^2+(\tfrac{3}{8}\frac{q^2}{p}+\frac{p^2}{36})y^2+qy+p=0,$$

$$\frac{\partial S}{\partial q} : \tfrac{3}{4}\frac{q}{p}y^2+y=0;\ q=-\tfrac{4}{3}\frac{p}{y}.$$

And the envelope is the ellipse

$$\frac{x^2}{\frac{-p}{3}}+\frac{y^2}{\frac{-12}{p}}=1.$$

Thus, *when the secondary conic is tangent to this ellipse the roots of the corresponding biquadratic equation form an harmonic range.* (See example, Fig. IV.)



$4x^4-24x^2+8x+3=0$

IV. Scale, $\frac{7}{32}$ in.=unity.

§ 3. The Quintic.

The roots of the general quintic

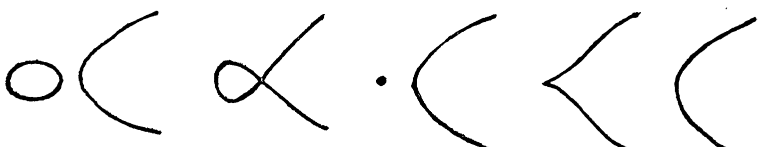
$$x^5+px^3+qx^2+rx+s=0 \tag{6}$$

are the abscissas of the five finite points of intersection of the curves

$$P : xy-1=0,$$
$$S : x^2+sy^3+ry^2+qy+p=0.$$

The second is a Newton divergent parabola.

There are five non-projective classes of the latter curves, as follows:



1 2 3 4 5

1. $sx^2=(y-\alpha)(y-\beta)(y-\gamma)$; all real factors, and $\alpha\neq\beta\neq\gamma$.
2. $sx^2=(y-\alpha)(y-\beta)^2$, $\alpha<\beta$; all real factors.
3. $sx^2=(y-\beta)^2(y-\alpha)$, $\beta>\alpha$; all real factors.
4. $sx^2=(y-\beta)^3$; three coincident roots.
5. $sx^2=(y^2+ay+b)(x-c)$; two imaginary roots.

Example: Construct the roots of the quintic equation

$$x^5-21x^3+50x^2-12x-8=0.$$

The secondary curve is

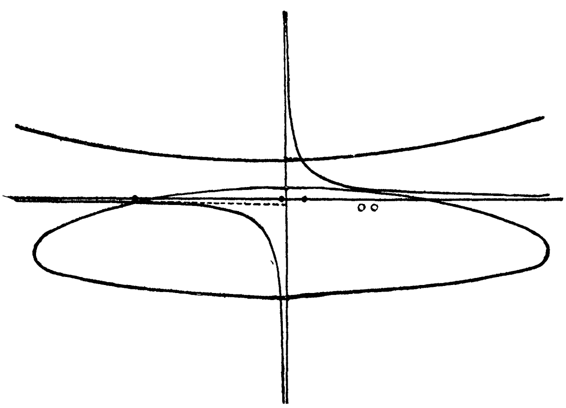
$$x^2-8y^3-12y^2+50y-21=0.$$

That is

$$\frac{x^2}{8}=(y-\tfrac{1}{2})(y-\tfrac{3}{2})(y+\tfrac{7}{2}).$$

Hence the secondary cubic is of type 1. From figure VI we have $r_1=.64+$, r_2 and r_3 imaginary, $r_4=-.3+$, $r_5=-5.4+$.

It is evident that the only classes of the Newtonian parabolas that can intersect the primary hyperbola in five real points are 1 and 2. Hence a necessary condition that all of the roots of a quintic (6) be real is that it be possible to resolve



VI. Scale, $\frac{3}{16}$ in.=unity.

$$sy^3+ry^2+qy+p=0$$

into real factors. That is, that the following relation hold:

(7) $$(9ps-qr)^2-4(3qs-r^2)(3pr-q^2)\leq 0.$$

This condition, though necessary, is not sufficient.

§ 4. A Reducible Case of the Sextic.

The sextic

(8) $$x^6+px^4+qx^3+rx^2+sx+t=0$$

is equivalent to the simultaneous equations

$$xz-1=0;\ x^3+px+tz^3+sz^2+rz+q=0.$$

In these z may be transformed linearly, taking

$$z=t^{-\frac{1}{3}}y+h$$

where h is a root of the cubic

(9) $$th^3+sh^2+rh+q=0.$$

This gives for the primary and secondary curves

$$P:\ xy+t^{\frac{1}{3}}hx-t^{\frac{1}{3}}=0,$$
$$S:\ x^3+px+y^3+uy^2+vy=0,$$

where $$u=t^{-\frac{2}{3}}(3ht+s);\ v=t^{-\frac{1}{3}}(3th^2+2sh+r).$$

The secondary cubic passes through the origin and has the asymptote

$$a_1:\ y=-x-\tfrac{1}{3}u.$$

Now S may degenerate into a straight line and a conic. In fact S degenerates if, and only if, one of the two following conditions is fulfilled:

$$d_2:\ v=p;\ u=0,$$
$$d_2:\ u=3\sqrt{-p};\ v=-2p\ (p \text{ necessarily negative}).$$

If $v\neq p$ and $u=0$, the curve S has the origin for a center of symmetry. All these cubics cross the fixed asymptote $y=-x$ at the origin. The origin is a point of inflexion, the inflexional tangent being

$$T:\ y=-\frac{p}{v}x.$$

Thus when $u=0$, *i. e.*, when

$$2s^3-9rst+27qt^2=0,$$

the variation of the secondary, corresponding to changes in the coefficients of the sextic (8), is characterized by the rotation of T around the origin, and when T coincides with the asymptote, S degenerates into straight line and ellipse

$$S:\ L.C\equiv(x+y)(x^2-xy+y^2+v)=0,$$

i. e., into the asymptote itself and the ellipse $C=0$, origin at center and axis coinciding with the asymptote.

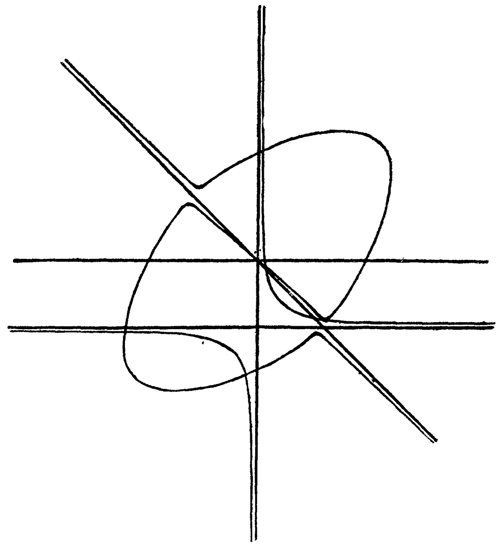
Figure [VII] shows the curves S and P when the sextic (8) is approaching

(10) $x^6-25.9x^4+50.6x^3-2x^2-4x+\frac{1}{3}=0.$

When condition d_2 holds we have

$$S:\ a_1C_1\equiv(x+y+\sqrt{-p})[x^2-xy+y^2+2\sqrt{(-p)}y-\sqrt{(-p)}x]$$

Thus S degenerates into the asymptote, and the ellipse C_1 through the origin, center at the point where the asymptote crosses the y axis, axis coinciding with the asymptote.



VII. Scale, $\frac{1}{9}$ in.=unity.

The sextic equations corresponding to d_1, d_2 may be solved by solving the cubic (9), extracting the roots $\sqrt[3]{t}$, $\sqrt{-p}$, and solving a quadratic and a biquadratic. In fact the sextics corresponding to d_1, d_2 are respectively,

(11) $$(x^2-t^{\frac{1}{3}}h\,x+t^{\frac{1}{3}})[x^4+t^{\frac{1}{3}}h\,x^3+(t^{\frac{2}{3}}h^2-t^{\frac{1}{3}}+p)x^2-2t^{\frac{2}{3}}h\,x+t^{\frac{2}{3}}]=0.$$

(12) $$[x^2+(\sqrt{(-p)}-t^{\frac{1}{3}}h)x+t^{\frac{1}{3}}][x^4+(t^{\frac{1}{3}}h-\sqrt{-p})x^3$$
$$+(t^{\frac{2}{3}}h^2-2\sqrt{(-p)}t^{\frac{1}{3}}h-t^{\frac{1}{3}})x^2+(2\sqrt{(-p)}t^{\frac{1}{3}}-2t^{\frac{2}{3}}h)x+t^{\frac{2}{3}}]=0.$$

FINITE PLANES WITH LESS THAN EIGHT POINTS ON A LINE.

By C. R. MAC INNES, Princeton University.

We get a finite projective plane if we have a finite number of points arranged in sets, called lines, and subject to the following conditions:

1. If A and B are two distinct points there is one and only one line joining them.

2. If α and β are two distinct lines there is one and only one point common to both.

3. There are at least three points on each line.

It follows from these, that we must have the same number of points on all lines. For, having two lines α and β, we can take a point not on either and join it to all the points of α. Each of these joins must also cut β and we therefore have the points on α and β paired.

Also, there must be the same number of points on a line as there are lines through a point. For, having any point A, take a line β not passing through A; join A to each of the points on β. Each point of β has a line through A and each line through A has a point on β.

In short, then, if we have $n+1$ points on a line, we have $n+1$ lines through a point; n^2+n+1 points in the plane, and n^2+n+1 lines in the plane.

Thus, if we have only three points on a line, we have seven points altogether. Denoting these by 0, 1, 2, ..., 6, we get the lines by the following cyclic scheme:

$$\begin{array}{l} 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6; \\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 0; \\ 3,\ 4,\ 5,\ 6,\ 0,\ 1,\ 2; \end{array}$$

the points in the columns being on a line.

If we leave out one line, we have a plane that might be called a finite Euclidean plane. In this, any two points determine a line, and through any point one and only one line may be drawn not meeting a given line. We have n points on each line, $n+1$ lines through each point, and the lines "parallel" in sets of n. An example of this for $n=3$ is the following: Denote a point by a_{ij} and write the points in the form

$$\begin{array}{lll} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}. \end{array}$$

Then the rows, columns, and terms in the ordinary determinant expansion give the twelve lines of the plane.

The existence* of such planes has been proved for the case $n=p^m$, p being prime and m any integer. It has also been proved that there is only one distinct type in each of the cases $n=2$, 3, and 4. Our present problem is for $n=5$ and $n=6$.

Having twenty-five points we are to get all possible Euclidean planes built from them. We may choose any set of parallels and write the points in rows, each row being the five points on one of the lines chosen. By properly choosing the order of the points in the rows, we may fix the columns to give another set of parallels. Finally, by properly choosing the order of the rows we may put into the main diagonal the points of some other line. We now have the scheme:

1.1	1.2	1.3	1.4	1.5
2.1	2.2	2.3	2.4	2.5
3.1	3.2	3.3	3.4	3.5
4.1	4.2	4.3	4.4	4.5
5.1	5.2	5.3	5.4	5.5

in which eleven lines are given by the rows, columns, and main diagonal. We know three of the lines through each point of the main diagonal; to find the others. The only possibilities for lines through 1.1 are

1.1	3.2	2.3	5.4	4.5	
		4.3	5.4	2.5	
		5.3	2.4	4.5	
. .	. .	. .	. .	. .	
	4.2	2.3	5.4	3.5	
		5.3	2.4	3.5	(A).
			3.4	2.5	
. .	. .	. .	. .	. .	
	5.2	2.3	3.4	4.5	
		4.3	2.4	3.5	
			3.4	2.5	

Of these we must pick three; they must be one of the following sets:

1.1	3.2	2.3	5.4	4.5	
	4.2	5.3	2.4	3.5	
	5.2	4.3	3.4	2.5	
. .	. .	. .	. .	. .	
	3.2	2.3	5.4	4.5	
	4.2	5.3	3.4	2.5	
	5.2	4.3	2.4	3.5	
. .	. .	. .	. .	. .	(B).

*Veblen and Bussey, *Transactions Mathematical Society*, April, 1906.

3.2	4.3	5.4	2.5
4.2	5.3	2.4	3.5
5.2	2.3	3.4	4.5
. .	. .	. .	. .
3.2	5.3	2.4	4.5
4.2	2.3	5.4	3.5
5.2	4.3	3.4	2.5

Doing the corresponding thing for the other points of the main diagonal, we have four other tables matching (B). We have now to pick a set of three from each table so that the fundamental assumptions are satisfied. This can be done in six ways. This gives twenty-six lines; the remaining four are parallel to the diagonal and can be written down immediately as they are determined uniquely by the others. Apparently there are six planes in this case; they can, however, be brought into a one-to-one correspondence, and again we have only one type. The method of establishing the correspondence analytically is not without interest.

We have denoted points by two numbers. If we think of these as coordinates of the points, we find that one of the six planes is such that the points of a line satisfy a first degree equation, using modulus 5 in the algebra. Thus $x=k$ gives the rows, $x-y=0$ the diagonal, and $x-y=k$ the lines parallel to the diagonal. And so on. The lines of this plane are, of course, interchanged by linear substitutions, of which there are 12,000. If to the plane we apply the transformations

$$x=3x_1^3+3\ldots(1),$$
$$x=3x_2^3+3x_2^2+x_2-1\ldots(2),$$
$$x=4x_3^3+2x_3^2+2x_3+3\ldots(3),$$
$$x=x_4^3-x_4^2+2x_4-1\ldots(4),$$
$$x=x_5^3+2x_5^2+3x_5\ldots(5).$$

y being subject to transformations of the same form, we get the five other planes. These transformations, though not linear, are birational, (1) and (2) being inverse to each other, and each of the remaining three being its own inverse. These five, with the identity, form a group.

If we consider the projective plane, it will be unaltered by linear fractional transformations. These form the simple group L.F(3.5) of order 372,000. This order is 31 times as large as in the Euclidean plane, as there are 31 lines, any one of which might be neglected to give the Euclidean plane from the projective one.

If we now try a similar thing for a plane with 36 points, the work proceeds as before till we have written down all the possibilities for lines through (1, 1), (2, 2), and (3, 3). The others need not be considered.

From the lines through (1, 1) we choose a set of four; to these add four through (2, 2) agreeing with them. This can be done in a number of ways. To the eight so found, it is impossible to add four through (3, 3) so that the fundamental assumptions hold. The first assumption breaks down, no matter what combination be tried.

There is therefore no Euclidean plane with 36 points; and consequently no projective plane with only seven points on a line. This includes the result for $n=7$ given in the answer to problem 142, p. 108, Vol. XIV. It is also the result given by Dr. F. H. Safford in answer to problem 132, p. 215, Vol. XIII.

ON CONSTRUCTING A CUBE HAVING A GIVEN RATIO TO A GIVEN CUBE.*

By R. D. CARMICHAEL.

The semi-cubical parabola is capable of a beautiful application to the problem of finding a cube having a given ratio to a given cube. The object of this paper is to give a method of constructing this curve by continuous motion, to apply the locus to the above problem, and also to show how to construct a line numerically equal in length to the cube root of a given line.

The equation of this curve,

(1) $$x^3=py^2,$$

when transformed to polar coordinates with the axis of x as the polar axis, may readily be reduced to

(2) $$\rho\cos\theta=p\tan^2\theta.$$

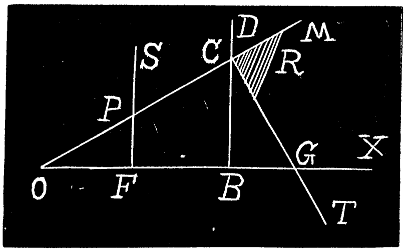


Fig. 1.

To find a construction of the curve by continuous motion we proceed as follows: Let OX (Fig. 1) be the polar axis. Take a distance $OB=p$ and erect BD perpendicular to OX, cutting OM at C, the angle MOX being equal to θ. Draw CG perpendicular to OM at C, intersecting OX in G. From O lay off $OF=BG$, and draw FS perpendicular to OX, intersecting OM in P. P is a point on the semi-cubical parabola.

Proof. Since $OB=p$, $BC=p\tan\theta$. But, since OCG is a right triangle and CB is perpendicular to OG, $OB:BC::BC:BG$. Hence,

*Read before the American Mathematical Society, April, 1907.

(3) $$p.BG=BC^2=p^2\tan^2\theta;\ \text{or}\ BG=p\tan^2\theta.$$

But by the equation of the curve, $p\tan^2\theta=\rho\cos\theta$. But if $OP=\rho$, then $OF=\rho\cos\theta=p\tan^2\theta=BG$. Hence P is a point on the locus.

Now since $OF=BG$ and $OB=p$, $FG=p$. The following construction of the curve by continuous motion is then evident: Let OM and OX be two material lines pivoted at O, OX being fixed to the plane of the paper. Let the material right triangle R, with the material straight line CG attached to it, be fastened to OM so that it may slide freely along OM. Let another material line $FG=p$ be attached to OX so as to slide along it and at the same time let CT pass through a ring at G of FG so as to slide freely. BC is a material line fastened to the plane of the paper at P and perpendicular to OX, OB being equal to p. It passes through a ring at the vertex of the right angle in the triangle R. Finally, FS is a material line fastened to GF perpendicular to it at its extremity F. This intersects OM at some point P. Then let a ring pass around OM and FS at P and let a pencil be placed in this ring. As OM revolves about O the pencil at P will describe the semi-cubical parabola $x^3=py^2$.

We shall now apply this locus to the problem of finding a cube having to a given cube the given ratio of the line m to the line n. For this we shall employ the special case $p=1$. In Fig. 2 lay off $x=OH=$a side of the given cube. Draw $HQ=y$, Q being a point on the locus. Take $OB=-1$ and erect BC perpendicular to OX and cutting QR in C, QR being parallel to OX. Draw OM through C, and then CG perpendicular to OM at C and cutting OX in G. By the same reasoning as in finding equation (3) it follows that

$$BG=\overline{BC}^2,=\overline{QH}^2=y^2.$$

But $x^3=y^2$. Hence $BG=OH^3$.

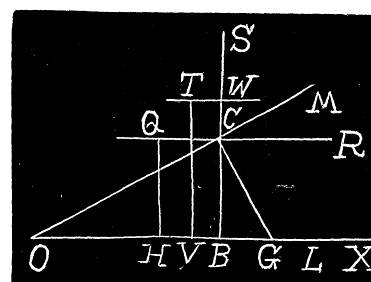


Fig. 2.

Now find a fourth proportional to m, n, BG, and from B lay off this line, say BL. Extend BC to S and on OL as a diameter describe a semi-circumference cutting BS at W and through W draw a line parallel to OX and cutting the curve in T. From T let fall a perpendicular to OX at V. Then we may repeat the previous reasoning to show that $OV^3=BL$. And therefore, since $OH^3=BG$ and $m:n::BG:BL$, $OH^3:OV^3::m:n$.

We have seen above that $OH^3=BG$, $OH=\sqrt[3]{BG}$. We may thus construct a line numerically equal in length to the cube root of any other line; or, conversely, we may construct a line numerically equal to the cube of a given line.

Remark. It will of course be observed that the problem here solved is a generalization of that of the duplication of the cube, renowned from

antiquity. By this method the ratio of the given cube to the required may be any capable of being expressed as the ratio of two straight lines, and that the problem is not more difficult of solution for one ratio than for another.

Presbyterian College, Anniston, Ala.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

314. **Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.**

Find the area of the triangle bounded by the lines $l\alpha+m\beta+n\gamma=0$; $l'\alpha+m'\beta+n'\gamma=0$: $l''\alpha+m''\beta+n''\gamma=0$, where α stands for $x\cos\alpha+y\sin\alpha-p$, etc. [See Salmon's *Conic Sections*, 6th Ed.]

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa., and C. N. SCHMALL, 89 Columbia Street, New York City.

Let $A=a(mn'-m'n)+b(nl'-n'l)+c(lm'-l'm)$,

$B=a(mn''-m''n)+b(nl''-n''l)+c(lm''-l''m)$,

$C=a(m'n''-m''n')+b(n'l''-n''l')+c(l'm''-l''m')$.

Then the intersection of (1) and (2) is

$$\frac{\alpha_1}{mn'-m'n}=\frac{\beta_1}{nl'-n'l}=\frac{\gamma_1}{lm'-l'm}=\frac{2\triangle}{A}.$$

The intersection of (1) and (3) is

$$\frac{\alpha_2}{mn''-m''n}=\frac{\beta_2}{nl''-n''l}=\frac{\gamma_2}{lm''-l''m}=\frac{2\triangle}{B}.$$

The intersection of (2) and (3) is

$$\frac{\alpha_3}{m'n''-m''n'}=\frac{\beta_3}{n'l''-n''l'}=\frac{\gamma_3}{l'm''-l''m'}=\frac{2\triangle}{C}.$$

The area of the triangle $(\alpha_1, \beta_1, \gamma_1)$; $(\alpha_2, \beta_2, \gamma_3)$; $(\alpha_3, \beta_3, \gamma_3)$ is

$$\frac{abc}{8\triangle^2}[\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)+\beta_3(\gamma_1\alpha_2-\gamma_2\alpha_1)+\gamma_3(\alpha_1\beta_2-\alpha_2\beta_1)].$$

Hence the area of the required triangle is

$$\frac{\triangle\, abc[l(m'n''-m''n')+m(n'l''-l'n'')+n(l'm''-l''m')]^2}{ABC}.$$

Also solved by the Proposer.

315. Proposed by **ROBERT E. MORITZ, Ph. D., University of Washington.**

Given the area of the segment of a circle of given radius to find the length of the chord.

Solution by **G. B. M., ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.**

I. Let r=radius and $2x$=the length of the chord. Also let A=arc of segment. Then

$$\tfrac{4}{3}x[r-\sqrt{(r^2-x^2)}\,[+\frac{[r-\sqrt{(r^2-x^2)}]^3}{4x}=A.$$

$$\therefore 169x^5+192r^2x^3-168Arx^2+144(A^2-r^4)x-288Ar^3=0.$$

If A and r are known, x can be found.

II. Let θ=angle of segment at center of circle. Then

$$\tfrac{1}{2}r^2(\theta-\sin\theta)=A,\quad x=r\sin\tfrac{1}{2}\theta.$$

By double position θ is found.

$$\text{III.}\quad r^2[\sin^{-1}\frac{x}{r}-\frac{x}{r^2}\sqrt{(r^2-x^2)}]=A.\quad \text{Let } \frac{x}{r}=z.$$

$\therefore r^2[\sin^{-1}z-z\sqrt{(1-z^2)}]=A.$
$\therefore \frac{2}{3}z^3+\frac{1}{5}z^5+\frac{3}{28}z^7+\frac{5}{72}z^9+\ldots=A/r^2.$
By reversion of series z is found, then $x=rz$.

316. Proposed by **J. STEWART GIBSON, Department of Physics, Wadleigh High School, New York City.**

Determine the locus of the vertices of parabolas described by particles thrown off from the circumference of a uniformly revolving wheel.

I. Solution by the **PROPOSER.**

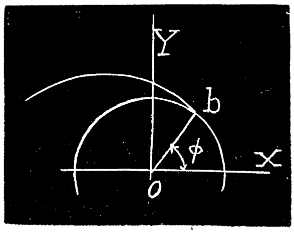
Let r=radius of circle, a=velocity of its periphery, ϕ=angular position of particle b at moment of projection, a_v=vertical component of initial velocity, and a_h=horizontal component of initial velocity. Then $a_v=a\cos\phi$. The height, y_1, to which the particle will rise is (since $h=v^2/2g$),

$$y_1=r\sin\phi+\frac{a^2\cos^2\phi}{2g}. \qquad (1)$$

The time of rise will be $t=\dfrac{a\cos\phi}{g}$. $a_h=a\sin\phi$.

$\therefore x_1$, the abscissa of the vertex of the parabola, is

$$x_1=-ta_h+r\cos\phi=r\cos\phi-\frac{a^2\sin\phi\cos\phi}{g}. \qquad (2)$$



Transposing and squaring (1),

$$r^2\sin^2\phi=y_1^2-\frac{y_1a^2\cos^2\phi}{g}+\frac{a^4\cos^4\phi}{4g^2};$$

whence $\cos\phi=\pm\sqrt{-\dfrac{2g(gr^2-y_1a^2}{a^4}\pm\sqrt{r^2-y_1^2+\dfrac{4g^2(gr^2-y_1a^2)^2}{a^8}}}$

and $\sin\phi=\sqrt{(1-\cos^2\phi)}=$

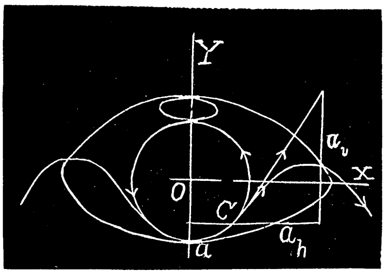
$$\pm\sqrt{1-\left[-\frac{2g(gr^2-y_1a^2)}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}\right]}.$$

Finally, the equation of the required locus is

$$x_1=\pm r\sqrt{-\frac{2g(gr^2-y_1a^2}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}}$$

$$\mp\left[\frac{a^2}{g}\sqrt{1-\left(-\frac{2g(gr^2-y_1a^2)}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}\right)}\right]$$

$$\times\left[\sqrt{-\frac{2g(gr^2-y_1a^2)}{a^4}\pm\sqrt{r^2-y_1^2+\frac{4g^2(gr^2-y_1a^2)^2}{a^8}}}\right].$$



The curve has the following peculiarities: It is symmetrical to the vertical axis; is tangent to the circumference at the inferior apex; and also at superior apex by the inclosed loop. Illustrations of the physical formation of the curve are oil drops from a pulley; mud particles from a carriage wheel; water drops from a revolving grindstone; sparks from revolving fireworks, such as "pin wheels," etc.

II. **Solution by G. W. GREENWOOD, Dunbar, Pa.**

Taking horizontal and vertical axes through the point of projection and in its plane, the path of a particle with initial velocity v, and making initially an angle θ with the vertical, is given by

$$x=v\sin\theta\, t,\ \ y=v\cos\theta\, t-\tfrac{1}{2}gt^2.$$

Eliminating t, we get as the equation to the path described,

$$x^2-\frac{2v^2x\sin\theta\cos\theta}{g}+\frac{2v^2\sin^2\theta y}{g}=0;\ i.\ e.,\ \left[x-\frac{v^2\sin 2\theta}{2g}\right]^2+\left[y-\frac{v^2\cos 2\theta}{2g}\right]$$

$$=\left[y-\frac{v^2}{2g}\right],$$

which is a parabola whose focus is

$$\frac{v^2\sin 2\theta}{2g},\ \frac{v^2\cos 2\theta}{2g}.$$

Now taking horizontal and vertical axes through the center of the wheel, and in its plane, the wheel being supposed to revolve clock-wise, the focus of the parabola described by a particle from the point $(-a\cos\theta,\ a\sin\theta)$, a being the radius of the wheel, is given by

$$x=\frac{v^2\sin 2\theta}{2g}-a\cos\theta,\ \ y=\frac{v^2\cos 2\theta}{2g}+a\sin\theta,$$

which is the equation of the required locus in terms of the parameter θ.

An excellent solution of this problem was received from G. B. M. Zerr.

CALCULUS.

241. **Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.**

Differentiate $y=1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\text{etc.}}}}}$

Solution by J. SCHEFFER, A. M., Hagerstown, Md.; FRANCIS RUST, Allegheny, Pa., and the PROPOSER.

The continued fraction is equivalent to $\frac{1}{2}+\surd(\frac{1}{4}+x)$.

Hence, $y=\frac{1}{2}+\surd(\frac{1}{4}+x)$, and $\dfrac{dy}{dx}=\dfrac{1}{\surd(1+4x)}$.

Also solved by G. B. M. Zerr, G. W. Greenwood, and A. H. Holmes.

242. Proposed by J. H. MEYER, S. J., Augusta, Ga.

A given sphere is to be formed into a solid composed of two equal cones on opposite sides of a common base, in such a manner that its surface may be the least possible. Find the dimensions of the solid, and compare its surface with that of the sphere.

Solution by A. H. HOLMES, Brunswick, Maine.

Of the cones into which the given sphere, radius R, is to be transformed, let x=radius of base, and y=altitude.

Then $\frac{2\pi x^2y}{3}=\frac{4\pi R^3}{3}$ or $x^2y=2R^3$ a minimum, or

$x^4+\frac{4R^6}{x^2}$==a minimum.

$\therefore 4x^3=\frac{8R^6}{x^3}$, and therefore $x=2^{\frac{1}{6}}R$ and $y=2^{\frac{2}{3}}R$.

Put S_1=surface of sphere, and S_2=surface of required solid.
Then $S_1:S_2=4:2^{\frac{4}{3}}\sqrt{3}$.

Also solved by G. B. M. Zerr.

MECHANICS.

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188. Proposed by H. L. ORCHARD, M. A., B. S.

Spherical bubbles of air are rising in water. Find the relation between radius and velocity.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let R=radius of bubble at surface of water, r=radius of bubble at start at bottom, δ=density of gas in bubble referred to water as unity, w= weight of one cubic inch of water in pounds, h=height of column of water equal to weight of one atmosphere, d=depth of water where bubble starts, v=velocity of bubble at distance s from starting point, bubble starting from rest, x=radius of bubble at distance s from starting point, f=acceleration.

$\therefore \frac{4}{3}\pi R^3w\delta$=weight of gas in pounds, $\frac{4}{3}\pi R^3w$=force, in pounds, impelling bubble upwards.

$$\therefore f=\frac{\frac{4}{3}\pi R^3w(1-\delta)g}{\frac{4}{3}\pi R^3w(1+\delta)}=\frac{(1-\delta)g}{1+\delta}. \quad \therefore v^2=2fs. \quad \text{Also } h+d:h+d-s=x^3:r^3.$$

$$\therefore s=\frac{(x^3-r^3)(h+d)}{x^3}. \qquad \therefore v^2=\frac{2f(x^3-r^3)(h+d)}{x^3}.$$

$$\therefore \frac{v^2x^3}{x^3-r^3}=2f(h+d)=\frac{2(1-\delta)\ (h+d)g}{1+\delta}$$

d can be found by either method in Vol. I, page 134.

202. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Three equal, uniform, similar rods AB, BC, CD, freely jointed at B and C, are hung from a point by two equal strings attached at A and D. Find the position of equilibrium.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

By symmetry, the strings, of length l, say, make equal angles with the vertical, as do also the rods AB and DC; denote these angles by θ and ϕ, respectively. The rod BC is horizontal. Denote the length of each rod by a, the weight by w, and the depth of the center of gravity of the system below the point of support by z, the strings being regarded as weightless.

$$z=[w(l\cos\theta+\tfrac{1}{2}a\cos\phi)+w(l\cos\theta+a\cos\phi)+w(l\cos\theta+\tfrac{1}{2}a\cos\phi)]/3w$$
$$=\tfrac{1}{3}[3l\cos\theta+2a\cos\phi].$$

For equilibrium, the value of z must be a maximum.

$$\therefore 0=3l\sin\theta\,d\theta+2a\sin\phi\,d\phi\ldots(1).$$

Also, by horizontal projection,

$$a=2l\sin\theta+2a\sin\phi\ldots(2).$$
$$\therefore 0=l\cos\theta\,d\theta+a\cos\phi\,d\phi\ldots(3).$$

$\therefore 3\tan\theta=2\tan\phi$ (by eliminating $d\theta$ and $d\phi$ from (1) and (2)). This equation, with equation (3), gives the position of equilibrium.

Also solved by G. B. M. Zerr and J. Scheffer.

AVERAGE AND PROBABILITY.

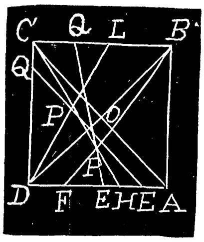
187. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let $ABCD$ be the given square, side a. P the randon point coordinates (u, v). On account of the symmetry of the square we will confine P to the triangle ADC. Let EQ be the random line through P, $m=\tan\theta=\tan QED$. For the area AOD, E must fall on HF to intersect opposite sides and on AH to intersect adjacent sides. For the area COD, E must fall on

HD to intersect opposite sides and on *AH* to intersect adjacent sides. For opposite sides $EQ=a\operatorname{cosec}\theta$. For adjacent sides $EQ=(a-u)\sec\theta+v\operatorname{cosec}\theta$. For opposite sides, the limits for *AOD* are, of v, 0 and $\frac{1}{2}a$; of u, $(a-v)$ and



v; of θ, $-\tan^{-1}\left(\dfrac{a-v}{u}\right)=\theta_2$ and $\tan^{-1}\left(\dfrac{a-v}{a-u}\right)=\theta_1$; for *COD*, the limits of u are $\frac{1}{2}a$ and a; of v, u and $(a-u)$; of θ, $-\tan^{-1}\left(\dfrac{v}{a-u}\right)=\theta_3$ and θ_1. For adjacent sides the the limits of u are 0 and a, of v, 0 and u, of θ, $\tan^{-1}\dfrac{v}{u}=\theta_4$ and θ_1. Let $\triangle$ be the required average length. Then we have for the denominator:

$$D=\int_0^{\frac{1}{2}a}\int_v^{a-v}\int_{\theta_2}^{\theta_1} dv\,du\,d\theta+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\int_{\theta_3}^{\theta_1} du\,dv\,d\theta+\int_0^{a}\int_0^{u}\int_{\theta_1}^{\theta_4} du\,dv\,d\theta$$

$$=\int_0^{\frac{1}{2}a}\int_v^{a-v}\left[\tan^{-1}\left(\frac{a-v}{a-u}\right)+\tan^{-1}\left(\frac{a-v}{u}\right)\right]dv\,du$$

$$+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\left[\tan^{-1}\left(\frac{a-v}{a-u}\right)+\tan^{-1}\left(\frac{v}{a-u}\right)\right]du\,dv$$

$$+\int_0^{a}\int_0^{u}\left[\tan^{-1}\left(\frac{v}{u}\right)-\tan^{-1}\left(\frac{a-v}{a-u}\right)\right]du\,dv$$

$$=\int_0^{\frac{1}{2}a}\left[2(a-v)\log\frac{(a-v)\sqrt{2}}{\sqrt{[(a-v)^2+v^2]}}+\frac{\pi}{\frac{1}{2}}(a-v)-2v\tan^{-1}\left(\frac{a-v}{v}\right)\right]dv$$

$$+\int_{\frac{1}{2}a}^{a}\left[2(a-u)\log\frac{(a-u)\sqrt{2}}{\sqrt{[(a-u)^2+u^2]}}-\frac{\pi}{2}(a-u)+2u\tan^{-1}\left(\frac{u}{a-u}\right)\right]du$$

$$+\int_0^{a}\left[(a-u)\log\frac{\sqrt{[(a-u)^2+a^2]}}{(a-u)\sqrt{2}}-u\log\sqrt{2}+\frac{\pi a}{4}-a\tan^{-1}\left(\frac{a}{a-u}\right)\right]du$$

$$=\tfrac{1}{4}a^2(\pi+2\log 2).$$

For the numerator we have

$$N=\int_0^{\frac{1}{2}a}\int_v^{a-v}\int_{\theta_2}^{\theta_1} a\operatorname{cosec}\theta\,dv\,du\,d\theta+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\int_{\theta_3}^{\theta_1} a\operatorname{cosec}\theta\,du\,dv\,d\theta$$

$$+\int_0^{a}\int_0^{u}\int_{\theta_1}^{\theta_4}[(a-u)\sec\theta+v\operatorname{cosec}\theta]du\,dv\,d\theta$$

$$=a\int_0^{\frac{1}{2}a}\int_v^{a-v}\log\left[\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a--u)}{\sqrt{[u^2+(a-u)^2]}+u}\right]dv\,du$$

$$+\,a\int_{\frac{1}{2}a}^{a}\int_{a-v}^{u}\log\left[\left(\frac{v}{a-v}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[(a-u)^2+v^2]}+a-u}\right)\right]du\,dv$$

$$-\int_0^a\int_0^u\left[(a-u)\log\left(\frac{u}{a-u}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}+(a-v)}{\sqrt{[u^2+v^2]}+v}\right)\right]$$

$$+v\log\left[\left(\frac{v}{a-v}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[u^2+v^2]}-u}\right)\right]du\,dv$$

$$=2a\int_0^{\frac{1}{2}a}\left[v\log\left(\frac{\sqrt{[(a-v)^2+v^2]}+v}{a--v}\right)-(a-v)\log(\sqrt{2}+1)+(a--v)\sqrt{2}\right.$$

$$\left.-\sqrt{[(a-v)^2+v^2]}\right]dv$$

$$+\,2a\int_{\frac{1}{2}a}^{a}\left[2(a-u)\log(\sqrt{2}+1)-(a-u)\log\{\sqrt{[(a-u)^2+u^2]}+u\}\right.$$

$$\left.-u\log\left(\frac{\sqrt{[(a-u)^2+u^2]}+(a-u)}{u}\right)+(a-u)\log)a-u)\right]du$$

$$+\,\tfrac{1}{2}\int_0^a\left[2au-u^2-a^2\sqrt{2}+(a-u)\sqrt{[(a-u)^2+a^2]}\right.$$

$$\left.-a^2\log\left(\frac{\sqrt{[(a-u)^2+a^2]}-(a-u)}{a(\sqrt{2}-1)}\right)\right]du$$

$$=\frac{a^3}{12}[9(\sqrt{2}+1)\log(\sqrt{2}+1)-4-5\sqrt{2}].$$

$$\therefore\ \Delta=\frac{N}{D}=\frac{a}{3}\left(\frac{9(\sqrt{2}+1)\log(\sqrt{2}+1)-4-5\sqrt{2}}{\pi+2\log 2}\right)$$

NOTE.—This solution differs from Mr. Heaton's in that Dr. Zerr assumes the lines to be secant lines, that is, lines whose extremities lie on the sides of the square. Mr. Heaton intended the lines to be radial lines, that is, lines the extremities of which are the random point and points on the sides of the square. In both solutions, the same law of distribution has been assumed. ED. F.

188. Proposed by **J. EDWARD SANDERS**, Reinersville, Ohio.

Find the average length of a hole at random through a given (a) sphere, (b) cube.

Solution by **G. B. M., ZERR, A. M., Ph. D.**, 4243 Girard Avenue, Philadelphia, Pa.

(a) Let one point remain fixed and be the origin. Then $x^2+y^2+z^2=2ax=d^2$ is the equation to the sphere of radius a.

Let $\sqrt{[2ax-x^2-y^2]}=z'$, $\sqrt{[2ax-x^2]}=y'$, $\triangle$=average length.

$$\therefore \triangle=\frac{\int_0^{2a}\int_0^{y'}\int_0^{z'} d\,dx\,dy\,dz}{\int_0^{2a}\int_0^{y'}\int_0^{z'} dx\,dy\,dz}=\frac{\sqrt{(2a)}\int_0^{2a}\int_0^{y'}\sqrt{x}\sqrt{[2ax-x^2-y^2]}dx\,dy}{\int_0^{2a}\int_0^{y'}\sqrt{[2ax-x^2-y^2]}dx\,dy}$$

$$=\frac{\sqrt{[2a]}\int_0^{2a}[2ax-x^2]\sqrt{x}\,dx}{\int_0^{2a}\sqrt{[2ax-x^2]}dx}=\frac{\frac{64a^4}{35}}{\frac{4a^3}{3}}=\frac{48a}{35}.$$

(b) Let the coordinates of ingress and egress of the hole be for opposite faces $(x, y, 0)$; (u, v, a), and for adjacent faces $(x, y, 0)$; and (a, v, z).

$\sqrt{[(x-u)^2+(y-v)^2+a^2]}=l$, $\sqrt{[(a-x)^2+(y-v)^2+z^2]}=l'$, average length$=p$.

$$\therefore p=\frac{\int_0^a\int_0^a\int_0^x\int_0^y l\,dx\,dy\,du\,dv+\int_0^a\int_0^a\int_0^a\int_0^y l'\,dx\,dy\,dz\,dv}{\int_0^a\int_0^a\int_0^x\int_0^y dx\,dy\,du\,dv+\int_0^a\int_0^a\int_0^a\int_0^y dx\,dy\,dz\,dv}$$

$$=\frac{4}{3a^4}\left[\int_0^a\int_0^a\int_0^x\int_0^y l\,dx\,dy\,du\,dv+\int_0^a\int_0^a\int_0^a\int_0^y l'\,dx\,dy\,dz\,dv\right.$$

$$=\frac{2}{3a^4}\int_0^a\int_0^a\int_0^x\left[y\sqrt{[(x-u)^2+y^2+a^2]}\right.$$

$$\left.+[a^2+(x-u)^2]\log\frac{y+\sqrt{[(x-u)^2+y^2+a^2]}}{\sqrt{[a^2+(x-u)^2]}}\right]dx\,dy\,du$$

$$+\frac{2}{3a^4}\int_0^a\int_0^a\int_0^a\left[y\sqrt{[(a-x)^2+y^2+z^2]}\right.$$

$$\left.+[z^2+(a-x)^2]\log\frac{y+\sqrt{[(a-x)^2+y^2+z^2]}}{\sqrt{[z^2+(a-x)^2]}}\right]dx\,dy\,dz$$

$$=\frac{2}{9a^4}\int_0^a\int_0^a\left[2xy\sqrt{[a^2+x^2+y^2]}+(3a^2y+y^3)\log\frac{x+\sqrt{[a^2+x^2+y^2]}}{\sqrt{[a^2+y^2]}}\right.$$

$$\left.+(3a^2x+x^3)\log\frac{y+\sqrt{[a^2+x^2+y^2]}}{\sqrt{[a^2+x^2]}}-2a^3\tan^{-1}\left(\frac{xy}{a\sqrt{[a^2+x^2+y^2]}}\right)\right]dx\,dy$$

$$+\frac{2}{9a^4}\int_0^a\int_0^a 2ay\surd[a^2+y^2+z^2]+(3z^2y+y^3)\log\frac{a+\surd[a^2+y^2+z^2]}{\surd[y^2+z^2]}$$

$$+(3az^2+a^3)\log\frac{y+\surd[a^2+y^2+z^2]}{\surd[a^2+z^2]}-2z^3\tan^{-1}\left(\frac{ay}{z\surd[a^2+y^2+z^2]}\right)\Bigg]dy\,dz$$

$$=\frac{1}{36a^4}\int_0^a\Big[4a^2y^2-2\pi a^4-2\pi y^4+4y(2a^2+y^2)^{\frac{3}{2}}+2y(a^2+y^2)^{\frac{3}{2}}$$

$$-6y^3\surd[a^2+y^2]-20a^4\tan^{-1}\left(\frac{y}{\surd[2a^2+y^2]}\right)+4y^4\tan^{-1}\left(\frac{y\surd[2a^2+y^2]}{a^2}\right)$$

$$+4a^4\tan^{-1}\left(\frac{\surd[2a^2+y^2]}{y}\right)+3a^4\log2+8(5a^3y+3ay^3)\log\frac{a+\surd[2a^2+y^2]}{\surd[a^2+y^2]}$$

$$+2a^2(3a^2+4y^2)\log\frac{y+\surd[a^2+y^2]}{a\surd2}+8(3a^4-a^2y^2)\log\frac{y+\surd[2a^2+y^2]}{a\surd2}\Big]dy$$

$$=\frac{a}{1620}[286-186\pi+464\surd2-372\surd3+768\log(1+\surd2)+1752\log(1+\surd3)$$

$$-936\log2]=\tfrac{25}{27}a,\text{ nearly.}$$

MISCELLANEOUS.

166. **Proposed by F. H. SAFFORD, Ph. D., The University of Pennsylvania.**

Several equal rectangular boxes are placed in a row with uniform intervals between the boxes and a passageway along one side of the row. Find the least width of the passageway permitting a box to be removed from the row without moving adjacent boxes. This problem arose during the construction of a room for storage batteries.

Solution by the PROPOSER.

Length of box $=PQ=a$; $\angle DAY=\phi=\cos^{-1}(b/c)$. Breadth of box $=PS=b$; $\angle RAY=\theta$. Space between boxes $=AB=c$; $OA=l=c\cot\phi$. Passage in front of boxes $=PW=h$; $OB=m=c\csc\phi$; $AR=n=c\csc\theta-b\cot\theta$.

The complete solution of this problem leads to seven cases corresponding to the possible relations between a, b, c. The desired solution is $h=PW$, and unless otherwise stated; $\phi\neq0$. The general cases are 1°, $a>c\csc\phi$; 2°, $c\csc\phi>a>c$; 3°, $c>a>c\sin\phi$; 4°, $c\sin\phi>a>0$.

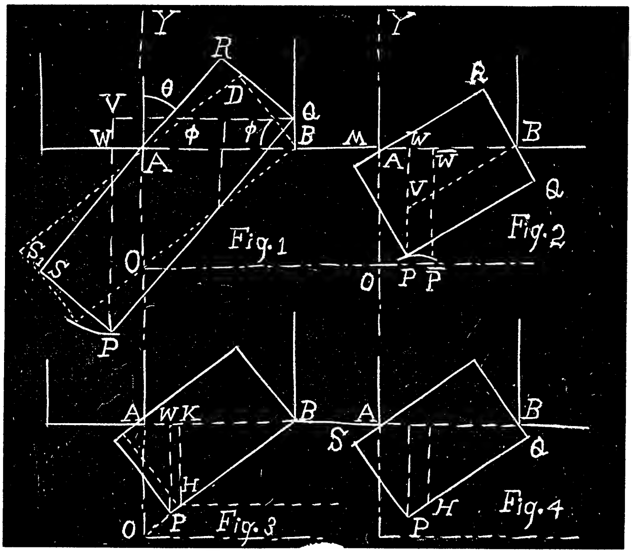
In case 1°, Fig. 1, the box remains partly in the "space" when PW is a maximum. The axes and the auxiliary constant ϕ are determined by the position of the box when Q coincides with B, and the minimum value of y corresponds to the desired value of PW. The path of P is given by

$$x=c-a\sin\theta,\ \ y=l+(n-a)\cos\theta-b\sin\theta, \tag{1}$$

in which θ is an auxiliary parameter. The minimum value of y is given by the solution of the transcendental equation

$$c=a\sin^3\theta+b\cos\theta. \tag{2}$$

The instantaneous center of motion of PO is at V, the intersection of the horizontal through Q and the normal to PQ through A. When the box is moving out of the space, V moves toward the left and P toward the right, the minimum y occuring when P crosses the vertical through V. This is the position shown in Fig. 1, from which equation (2) may be verified.



The solution for case 1° is

$$h=l-y=a\cos\theta+b\csc\theta-c\cot\theta, \tag{3}$$

in which θ is the solution of (2).

Fig. 1 shows that h is less than a, so there would be no advantage in withdrawing the box without turning, and then moving it sidewise.

Case 1° is particularly important because the critical point is not at the position BS_1. The limiting case between 1° and 2° occurs when $a=c\csc\phi=OB$, and the solution is then $h=a\cos\phi=ab/c$.

In case 2°, $c\csc\phi>a>c$, the ordinate of P is still decreasing when Q reaches B, as may be shown by the derivative of the second equation in (1) viz.,

$$\frac{dy}{d\theta}=\frac{-1}{\sin^2\theta}(c-a\sin^3\theta-b\cos\theta). \tag{4}$$

Hence the path of P must be studied when SR and QR have sliding contact with A and B, respectively. From Fig. 2,

$$\left.\begin{aligned} x&=b\cos\theta-a\sin\theta+c\sin^2\theta,\\ y&=l-a\cos\theta-b\sin\theta+c\sin\theta\cos\theta, \end{aligned}\right\} \tag{5}$$

$$dy/d\theta = a\sin\theta - c\sin^2\theta + c\cos^2\theta - b\cos\theta = MA - BN. \quad (6)$$

Thus when Q has passed B, the ordinate of P changes and b becomes an increasing one. When a is a little greater than c, P may be in the same vertical as V, the instantaneous center, as seen in Fig. 2, where it is evident that (6) vanishes. This fact does not lead to a solution because P is here at a maximum, so that the provisional solution comes when Q coincides with B, *i. e.*, $\overline{PW}$, Fig. 2.

On PQ let $HQ = \bar{a}$ be laid off so that

$$c > \bar{a} > c\sin\phi, \quad (7)$$

then from Fig. 3,

$$HK = \bar{a}b/c, \quad (8)$$

and it is to be noticed that a further change of position, as shown in Fig. 4, gives by computation, the same value for HK. At the same time (6) shows that during this motion from Fig. 3 to Fig. 4, the ordinate of P increases and then decreases. Since the slope of PQ is less in the latter position, it follows that the ordinate of P is greater in Fig. 4.

If instead of the restriction in (7), $\bar{a}$ be so chosen that

$$c\csc\phi > \bar{a} > c, \quad (9)$$

it follows from (6), since MA does not vanish when R reaches B, that the ordinate of P is still decreasing, hence from (8), with a replacing $\bar{a}$ so as to apply to case 2°,

$$h = PW = ab/c. \quad (10)$$

Thus (10) is the solution for case 2° and gives a value greater than b. In the limiting case between 2° and 3°, *i. e.*, $a = c$, the value is precisely b.

For case 3°, $c > a > c\sin\phi$, Fig. 4 is available if the left boundary is AH instead of SP, so that (8) gives

$$h = HK = ab/c,$$

a result less than b. But a rotation about A must follow, making AH vertical, and now the solution is $h = b$. The limiting case between 3° and 4°, *i. e.*, $a = c\sin\phi$, is that in which the box can just rotate in the space. This case may be considered with case 4°, $c\sin\phi > a$, and the joint solution is $h = b$, as complete rotation is possible.

In the last and trivial case, where $\phi = 0$, b is equal to c, and the solution is $h = a$, because the box cannot be rotated in the space, but must be withdrawn by translation.

PROBLEMS FOR SOLUTION.

ALGEBRA.

289. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\left(\frac{n+1}{(n+1)^2-1}+\frac{1}{3}.\frac{n+3}{(n+3)^1-1}+\frac{1}{5}.\frac{n+5}{(n+5)^2-1}+\dots\right)$
$+\left(\frac{2}{3}.\frac{1}{n+2}+\frac{4}{15}.\frac{1}{n+4}+\frac{6}{35}.\frac{1}{n+6}+\dots\right)=\frac{n-1}{(n-1)^2-1}+\frac{1}{3}.\frac{n-3}{(n-3)^2-1}$
$+\frac{1}{5}.\frac{n-5}{(n-5)^2-1}+\dots+\frac{1}{l}.\frac{n-l}{(n-l)^2-1}$, n being any odd integer greater than 1 and $l=n-2$.

290. Proposed by G. I. HOPKINS, M. A., Manchester, New Hampshire.

A and B are 45 miles apart and travel towards each other. A goes one mile the first day, three miles the second day, five miles the third day, and so on. B goes two miles the first day, four the second, six the third, and so on. In how many days will they meet? What interpretation is to be placed upon the negative value of n?

GEOMETRY.

221. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Texas.

ABC is an isosceles triangle. Through any point P in its plane draw a line $PSRT$ cutting the sides AC, CB, AB in the points S, R, and T, respectively (R between B and C), so that the segments CS and BT shall be equal.

222. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Find all surfaces such that the normal lengths intercepted by the three coordinate planes are in constant ratios for all points.

223. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

S, S' are the foci of two co-vertical parabolas A and B, the axes of which are at right angles. Draw the circle K on SS' as diameter. K is cut in D and E by a straight line parallel to the axis of A such that S' lies midway between it and that axis. Show that the lines $S'D$, $S'E$ are parallel to the two tangents to A which are normals to B.

CALCULUS.

245. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove or disprove:

$$\int_0^\infty \frac{dx}{\surd(x^2-1)(k^2x^2-1)}=2F^I(k)+\surd(-1).F^I[\surd(1-k^2)],$$

Legendre's notation, $0<k<1$.

246. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Derive Taylor's Series by the use of the formula for successive integration by parts, and nothing else.

MECHANICS.

207. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A portion of a parabola is bounded by the curve, the axis and an ordinate. A circle is inscribed to the figure which is regarded as a plane lamina. The area of the inscribed circle is now punched out. Find the centroid of what is left.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

148. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find all the multiply perfect numbers of n different prime factors and of multiplicity $n-1$.

AVERAGE AND PROBABILITY.

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random lines cut a given circle. What is the chance that they intersect within the circle?

192. Proposed by A. H. HOLMES, Brunswick, Maine.

In the game of baccarat the dealer and each side of the table have two or three cards. The object is to get as near nine as possible, and tens and court cards do not count. If the two first cards dealt do not together amount to five, the player asks for another. If above five he does not. When the two cards amount to exactly five would the chances of the hand be bettered or diminished by drawing a third card, and how much?

MISCELLANEOUS.

173. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

If n is odd, prove the following: $\pm 1=[(-1)^{1/n}+(-1)^{-(1/n)}][(-1)^{2/n}+(-1)^{-(2/n)}][(-1)^{3/n}+(-1)^{-(3/n)}]\ldots[(-1)^{(n-1)/2n}+(-1)^{-(n-1)/2n}]$ $\pm\sqrt{n}(-1)^{(n-1)/4}=[(-1)^{1/n}-(-1)^{-(1/n)}][-(-1)^{2/n}-(-1)^{-(2/n)}][(-1)^{3/n}-(-1)^{-(3/n)}]\ldots[(-1)^{(n-1)/2n}-(-1)^{-(n-1)/2n}]$.

174. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

By a linear transformation with integral coefficients modulo 2, reduce $\Sigma x_i^2+\Sigma x_ix_j$ $(i, j=1, \ldots, 2m; i<j)$ to a canonical form in which the variables are separated into pairs.

NOTES AND NEWS.

A regular meeting of the Southwestern Section of the American Mathematical Society will be held in the buildings of Washington University, St. Louis, on Saturday, November 30, 1907. S.

The next annual meeting of the American Association for the Advancement of Science will be held in Chicago during the Christmas Holidays. Two joint sessions of Sections A and O, Mathematics and Engineering, with the Chicago Section of the American Mathematical Society will be held on Monday afternoon and Tuesday morning, December 30 and 31. The subject for discussion is: "The Teaching of Mathematics to Students of Engineering." Prominent engineers and mathematicians will take part in the program. S.

The sixth annual meeting of the Central Association of Science and Mathematics Teachers will be held in the building of the McKinley High School at St. Louis, Mo., November 29 and 30, 1907. The Mathematics Section will hold sessions Friday afternoon and Saturday morning, at which the reports of two important committees will be presented, one on the teaching of algebra, and the other on the teaching of geometry. The leading speakers in the discussion of these reports will be Professor Florian Cajori, Colorado College; Professor E. R. Hedrick, University of Missouri; Professor G. B. Halsted, Greeley, Colo.; Professor G. C. Shutts, Whitewater, Wis.; W. W. Hart, Shortridge High School, Indianapolis; and C. W. Newhall, Shattuck School, Faribault, Minn. S.

Daniel A. Murray, Ph. D., has recently accepted the chair of Mathematics in McGill University. The title of the latest of his many popular college text books has been changed from "Practical Mathematics" to "Essentials of Trigonometry." OLIPHANT.

Readers of the MONTHLY may secure a year's subscription to the *Technical World Magazine* (regular price, $1.50) and Finkel's *Mathematical Solution Book* (regular price, $2.00), for $2.25 for the magazine and book. For $4.00, one year's subscription to the MONTHLY will be included. *The Cosmopolitan* and the MONTHLY for one year for $2.70.

BOOKS.

A First Course in the Differential and Integral Calculus. By William F. Osgood, Ph. D., Professor of Mathematics in Harvard University. 12mo. Cloth, xv+423 pages. Price, $2.00, net. New York: The Macmillan Co.

Important features, among others, in the treatment of the Calculus as carried out in

this work are, the simplicity, the clearness, and the directness with which the principles underlying the Calculus are set forth and the splendid applications and illustrations of these principles in the solution of problems in physics and mechanics. Since the ideas underlying the Calculus are nowhere brought out more clearly than in the application of its principles to the study of curves and surfaces, in Mechanics, and in definite integrals with their applications to Geometry, Physics, and Astronomy, these subjects are taken up at an earlier stage than is usually customary. Thus, for example, *curvature* is taken up in chapter VII, p. 134; *definite integrals*, Chapter IX, p. 153, and *mechanics*, Chapter X, p. 190. Chapter XVIII deals with double integrals, and Chapter XIX with triple integrals. Here the author uses a notation which should be followed by all writers on the subject, viz., for example, for

$$\int\int\int f(x,\ y,\ z)\,dx\,dy\,dz,$$

the notation

$$\int dx \int dy \int f(x,\ y,\ z)\,dz$$

is used. By this notation there is no ambiguity as to the order of integration. The book contains many valuable features too numerous to mention in the brief space at our disposal. It is very attractively printed and bound, and the selection of problems is most commendable. F.

High School Algebra. Elementary Course. By H. E. Slaught, Ph. D., Assistant Professor of Mathematics in the University of Chicago, and N. J. Lennes, M. S., Instructor in Mathematics in the Wendell Philips High School. 8vo. Cloth, xii+297 pages. Chicago: Allyn and Bacon.

As stated in the preface, the important features of this text-book are: (1) Algebra is here vitally and persistently connected with arithmetic; (2) the enunciation of the principles of algebra in eighteen short sentences; (3) the solution of problems rather than the construction of purely theoretical doctrine as an end in itself; and (4) the determination of the order of the topics and the inclusion of the order of the topics, and the inclusion and exclusion of subject matter by the main purpose of the course itself.

The book is based on true scientific and pedagogical principles. Great care is taken in laying the foundation of the subject. The problems are drawn, for the most part, from the experiences of every-day life and are of a nature easily within the comprehension of the beginner. From a pedagogical point of view the book is all that can be desired. It seems that all pedagogical requirements have been satisfied not only in this work but perhaps in several of its predecessors, and that the attention of teachers of mathematics in the higher institutions of learning be now directed to, what seems to me, to be the true cause of the lamentable state of elementary mathematical teaching in this country. The cause of poor teaching is not so much the lack of teachable text-books in the hands of the pupils as the lack of teachable teachers into whose hands the pupils are committed. Is it not a fact that the teaching of Algebra and Geometry in the great majority of high schools and academies is intrusted to the merest arithmetical tyros, teachers whose thoughts in regard to mathematics are as dark and confused as are those of a savage respecting the laws of the universe. The most noteworthy progress in the teaching of elementary mathematics will be obtained when teachers who have no more interest in mathematics than to make the perfunctory teaching of it a means to gain a livelihood are crowded to the rear and their places taken by the real, earnest, enthusiastic, and enlightened teacher of mathematics. F.

Plane and Solid Geometry. By Isaac Newton Failor, Principal of the Richmond Hill High School, New York City. 12mo., 420 pages. $1.25, net. New York: The Century Co.

The author has aimed to present to the educational public a work on Geometry that should be both teachable and practicable. In the earlier parts of the book most corollaries

are proved and references and postulates are quoted in full. This is necessary in order to give the beginner a notion of what is required to be done. The demonstrations are so arranged that no page needs be turned to read them. A change which, to my mind, does not add to the attractiveness of the book, is that the theorems are set in ordinary long primer type instead of in italics or black-faced type. The book contains a very large collection of exercises well suited to call out the powers of the student.

The publishers have done all that is possible to make the mechanical features of the book first class. F.

Text-book of Mechanics. By Louis A. Martin, M. E. (Stevens), A. M. (Columbia), Assistant Professor of Mathematics and Mechanics in Stevens Institute of Technology. Vol. II, Kinematics and Kinetics. 12mo. Cloth. xiv+214 pages. 91 figures. Price, $1.50, net. New York: John Wiley and Sons.

This volume completes the author's elementary course in Mechanics, the intention of which course is to prepare the student for courses in Applied Mechanics, and to lay a solid foundation for the study of more difficult works. The study of this volnme requires a knowledge of Analytical Geometry and the Calculus. There are many exercises, the solution of which will enable the student to gauge his own knowledge of the subject as he pursues his course. The book is neatly printed and bound. F.

The Elements of Plane and Spherical Trigonometry. By Edwin S. Crawley, Ph. D., Thomas Scott Professor of Mathematics in the University of Pennsylvania. New and Revised Edition. Entirely rewritten. 8vo. Cloth, v+186 pages. Price, $1.25. Philadelphia: Published by the Author.

In this new edition, important changes and additions have been made. Of these, we note the addition of trigonometric equations and elimination, trigonometric series, and hyperbolic functions. Also some additions have been made in the discussions of lines and circles. Thus, some properties of the nine-points circle have been introduced and the determination of the Brocard points. The book concludes with a prief application of trigonometry to Astronomy. The typography of the book is first class and the binding and paper are excellent. F.

Computation and Mensuration. By P. A. Lambert, M. A., Professor of Mathematics in Lehigh University. 8vo. Cloth, ix+92 pages. Price, $0.80. New York: The Macmillan Co.

This work is divided into ten chapters, the first of which deals with Approximate Computation; the second, with Graphic Computation; the third, with the Method of Coordinates; the fourth, with Volumes of Solids Bounded by Planes; the fifth, Computation and Use of Trigonometric Functions; the sixth, with Computation and Use of Logarithms; the seventh, with Limits; the eighth, with Graphic Algebra; the ninth, with Areas Bounded by Curves; and the tenth, with Volumes of Solids.

The aim of the work is to give the student a training in the application of the knowledge gained in the secondary school mathematics, and is intended to come at the close of the secondary school course or at the beginning of the college course.

The work is well conceived and will, if properly used, serve to increase the student's power and enable him to carry on his work in the college with interest and pleasure. F.

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LIBRARY AIDS TO MATHEMATICAL STUDY.

By DR. G. A. MILLER, University of Illinois.

Herr Valentin of Berlin who has been working on a general mathematical bibliography for more than twenty years estimates that the total number of different mathematical works is about 35,000 and that about 95,000 mathematical articles have appeared in the various periodicals.* Morever, the amount of this literature is growing at an increasing rate of speed so that it appears likely that during the next forty years there will be a larger addition to the mathematical literature than the total amount which has appeared up to the present time. In fact, this is a very conservative estimate, since such a work as the *Jahrbuch der Fortschritte der Mathematik* chronicles annually about 2000 books and articles in pure mathematics in addition to a large number in closely allied subjects.

One of the first questions which confronts the student is the relative importance of periodic and non-periodic publications. In general it must be said that these supplement each other and that the advanced student needs both. As the books generally co-ordinate the results obtained by many different writers, broad views can usually be more easily obtained from books than from the separate articles, but these broad views are tinged by the peculiar bent of the author's mind and they naturally do not exhibit the clearness in detail which the student would have obtained by studying the authorities himself. This is especially true of the newer subjects where the number of books is comparatively small and where progress is generally so rapid that the books are deplorably behind the times.

The history of mathematics furnishes a good illustration of the point in question. When the first edition of the first three volumes of Cantor's *Vorlesungen über Geschichte der Mathematik* appeared, it was commonly regarded as authoritative even in nearly all of the details. It was to a large extent instrumental in arousing a more general interest in the history of mathematics and marked the beginning of an unusually active period in historical investigations, so that it is becoming much more difficult for one man

*Felix Mueller, *Bibliotheca Mathematica*, Vol. 7 (1907), p. 416.

to make use of all the available knowledge for a general history. Hence we find that the first volume of the third edition of this great work, which has appeared recently, and contains a large number of improvements over the preceding editions, has met with severe criticism,* and it appears that the student of mathematical history is compelled to get his knowledge largely from the journals if he wants to feel certain that he is not holding views which have been proved to be incorrect in well known literature.

Ever since mathematics has had a considerable literature both the investigator and the student have felt the need of better facilities to learn just what has been done. A very interesting discussion of this need is found among the earliest publications of the Royal Society of London. In the volume for 1681-2 Dr. Pell suggests "that the three following works be composed and published: 1. *Mathematical Pandects*, containing, as clearly, methodically, concisely, and ingeniously, as can be done, whatever may be collected, or deduced by way of corollary, from mathematical books and discoveries made before our time; quoting the most ancient authors in which they are found, and noting in all following authors where they have pilfered from others without acknowledgment; or, what is worse, have arrogated to themselves the inventions of others. By this means, that large library would be contracted into a narrower compass, to the great saving of labour, time, and expence, for those that come after. 2. *A Mathematical Compendium*, containing, in a concise manual, all the most useful tables, with precepts to show their application to the solution of problems, either of pure mathematics, or applied to other subjects. Finally, that we may not always be confined to books in this kind of learning, there should be contrived, 3. *The Self-sufficient Mathematician*, or an instruction to show how any mathematician, not averse to labour, may acquire so much skill, that without the aid of books or instruments he may accomplish the solution of any mathematical problem, and that as easily as another by only turning over books."

In answering some suggestions by the noted French mathematician, Mersenne, Dr. Pell makes the following interesting remarks in the same volume:† "Now the less I am pleased with these minute mathematicians, the more I should wish for a library of this kind, as being the only method of curing that licentious itch of scribbling. For these prating pretenders, ever trifing in a childish manner, while they would seem to accommodate themselves to the capacity of youth, may see that there are already too many who have compiled rudiments of this kind. And those who fondly aim at advancing the mathematical sciences by an infinity of new discoveries, when they see so many empty paradoxes, which have been condemned and ridiculed by the public, may take warning by the miscarriage of others. But especially the plagiaries, those pests of all true literature, will not have the

*Ref. G. Enestroem, *Bibliotheca Mathematica*, Vol. 7 (1907), p. 398.

†These extracts are interesting on account of the style and are illustrations of some of the earliest periodic mathematical literature.

impudence to vend, as their own, any old books, or any parts of them, which perhaps have not been printed more than once."

While these statements, coming to us through more than two centuries, do not fit to present conditions yet they exhibit the same yearning for convenient means to learn the known along some lines. Fortunately, an increasing number of mathematicians have been willing to devote their best energies to the work of making it easier for others to find out what is known about a particular subject. Although our best bibliographical works are far from perfect yet they are of an incalculable value for the advancement of knowledge. Some of these, like the *Juhrbuch der Fortschritte der Mathematik,* have been conducted by one or two men with such assistance as they could get from their colleagues; others, like the *Revue semestrielle des publications mathêmatiques* and the *Royal Society Catalogue,* are conducted by bodies of learned men; still others, like the *International catalogue of scientific literature,* are directed by international councils and supported by a large number of different countries.

One of the greatest advantages of such bibliographical works is that they enable the student to find quickly what has been done along a particular line. A classification which has been very extensively adopted is due to the international congress of mathematical bibliography held at Paris in 1889. It aims to give a very detailed classification of mathematical subjects but *does not consider the different methods employed in treating these subjects.* This classification is explained in a small volume entitled *Index du rêpertoire bibliographique des sciences mathêmatiques* published by Gauthier-Villars et Fils of Paris. Among many other places it has been adopted by the *Revue semestrielle* and by the *Bulletin of the American Mathematical Society.* The largest divisions of the entire subject of mathematics are denoted by capital letters of the Roman alphabet, subdivisions when necessary being denoted by exponents. Further successive subdivisions are indicated by number symbols, small Roman letters and small Greek letters. Hence this scheme provides for an almost endless division and even at the present time it sometimes enables one to obtain all the classified literature on a particular subject by looking over less than one thousandth part of the entire mathematical literature of the period.

From the above it is evident that the books on books are almost as important to the student as the original works themselves. In fact, most advanced students will probably use these books on books more frequently than any other equal number of volumes. In addition to the four great bibliographical works which have been mentioned the great encyclopedias (German and French) which are now in the process of publication, and the *Encyklopaedie der Elementar-Mathematik* by Weber and others, which was completed recently, are especially helpful to the student. Hagen's *Synopsis der hoeheren Mathematik* and Carr's *Synopsis of elementary results of pure mathematics* are also frequently very convenient. It is however not our

purpose to give a long list of bibliographical aids to the mathematical student. Such a list may be found in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 12 (1903), pp. 408-426. Our main object has been to convey an accurate idea in reference to the magnitude of the total mathematical literature and some of the aids to use this literature wisely in the better libraries.

THREE THEOREMS ON THE TRISECTION OF AN ACUTE ANGLE.

By J. SAMSONOFF, New York City.

THEOREM I. *The line DE (Fig. 1), which passes through the vertex A of one of the equal angles of an isosceles triangle ACB and intercepts on the line BC a part DC equal to the chord EC subtending the arc EC, which is drawn with radius AC from point A as a center, is the trisectorial line for the angle ABC.*

We have given the isosceles triangle ACB and the circumference FEC, which is drawn from the point A as a center and with AC as a radius, also the line DE which passes through A and intercepts on BC a part DC equal to the chord EC.

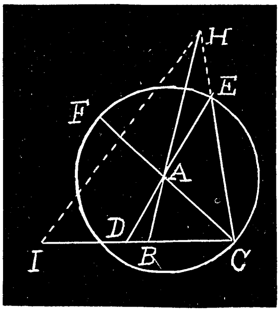


Fig. 1.

We are to prove that the line DE is the trisectorial line for $\angle ABC$, that is, $\angle DAD = \dfrac{\angle ABC}{3}$.

PROOF. Prolong the line CE until it intersects the line BA at the point H, and from H draw a parallel to ED, which will intersect the line DC at the point I.

Now, $\angle CAB$, the exterior of triangle HCA, is equal to $\angle ACH + \angle CHA$; $\angle ABC$, the exterior angle of triangle IHB, is equal to $\angle BIH + \angle IHB$; but triangle EDC is isosceles (by hypothesis), triangle HCI is isosceles (by construction, as HI is parallel to ED); and triangle CAE is isosceles (because $AC = AE$).

Therefore, $\angle ACH = \angle BIH$, and $\angle IHB = \angle CHA$. But $\angle IHB = \angle HAE$ (lines IH and DE are parallel by construction).

Therefore triangle AEH is an isosceles triangle and $AE = EH$. Hence, $\angle ABC = \angle DAB + \angle ADB = 3(\angle DAB)$, or line DE is the trisectorial line for $\angle ABC$.

THEOREM II. *The bisector CF of the angle at the vertex of an isosceles triangle ACB (Fig. 2), prolonged to the intersection with the trisectorial line DE, forms an isosceles triangle FEC.*

We have given the line CF bisecting $\angle ACB$, the vertical angle of an isosceles triangle ACB, and the trisectorial line DE, which is drawn through the vertex A of one of the equal angles of the same isosceles triangle ACB.

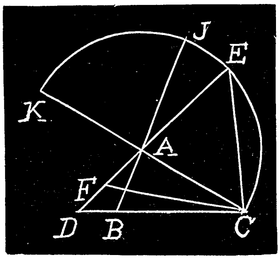


Fig. 2.

We are to prove that FEC is an isosceles triangle.

PROOF. Prom the point A as a center and with AC as a radius describe a semicircle $KJEC$. As the line DE is the trisectorial line (by hypothesis), then $EC=DC$ (according to Theorem I); therefore DCE is an isosceles triangle.

But CAE is also an isosceles triangle ($AC=AE$).

Hence as DCE and CAE have a common angle AEC, we have $\angle ACE=\angle EDC$.

Then $\angle EFC=\angle EDC+\angle FCD$ ($\angle EFC$ is the exterior angle of triangle DFC); and $\angle ECF=\angle ECA+\angle ACF$. But $\angle ECA=\angle EDC$ (by construction), and $\angle ACF=\angle FCD$ (by hypothesis), hence the triangle FEC is isosceles.

THEOREM III. *The chord DE (Fig. 3), which intersects the bisector AC of the angle at the vertex of an isosceles triangle DAB, forming the segment FE=BA, equal to the radius of the circumference, is the trisectorial line for the angle BDA.*

We have given the isosceles triangle DAB. From the point A as a center a semicircumference $DCBE$ is drawn with the radius AB. The line AC bisects $\angle BAD$, that is, $\angle BAC=\angle CAD$. The chord DE, which intersects AC at F, cuts off a segment $FE=AB$.

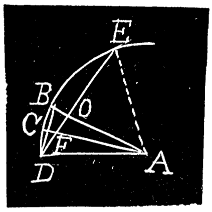


Fig. 3.

We are to prove that DE is the *trisectorial* line for $\angle BDA$, that is, $\angle BDE=\dfrac{\angle BDA}{3}$.

PROOF. Join points A and E. Since $AE=AD$ (the radii of the same circumference), the triangle DAE is isosceles. Therefore, $\angle EDA=\angle DEA$. But triangle FEA is also isosceles ($FE=BA=EA$ by hypothesis). Hence $\angle EAF=\angle EFA$.

But $\angle EFA=\angle ADF+\angle FAD$, and $\angle EAF=\angle EAO+\angle OAF$. And as $\angle OAF=\angle FAD$ (by hypothesis), we have $\angle EAO=\angle FDA=\angle OEA$.

Hence, triangle AOE is also isosceles. Now, as the measure of $\angle OAE$ is arc BE, and the measure of $\angle BDE$ is $\dfrac{\text{arc } BE}{2}$, therefore $\angle OAE$ or $\angle ODA=2(\angle BDE)$. That is, DE is the trisectorial line for $\angle BDA$.

PROBLEM. *To divide an acute angle into three egual parts by means of a graduated ruler and a compass.*

SOLUTION. Let $\angle BAC$, an acute angle (Fig. 4), be divided by line AD into three equal parts. Take C as a center and with AC as a radius circumscribe a semicircle ABD, intersecting AB and AD at points B and D.

Connect points B and D with the center C. Let BC intersect AD at point O.

Triangle ACD is isosceles ($AC=CD$). Therefore $\angle DAC=\angle ADC=2(\angle BAD)$.

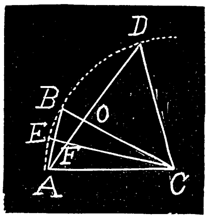


Fig. 4.

But triangle COD is also isosceles since the measure of $\angle BCD$ is arc BD; and the measure of $\angle BAD$ is one half of arc BD, making $BCD=2BAD=\angle DAC$. Therefore $\angle OCD=\angle ODC$.

Bisect $\angle BCA$, and let the bisector EC intersect AD at F. Now, triangle FDC is isosceles, for $DFC=\angle DAC+\angle FCA$; $\angle DCF=\angle DCO+\angle OCF$; $\angle DAC=\angle DCO$ as just proved, and $\angle OCF=\angle FCA$ by construction (we assumed EC is the bisector of $\angle BCA$).

Therefore $\angle DFC=\angle DCF$, and line $DF=DC$ equals the radius of the circle. From this analysis we come to the construction of the *trisectorial* line.

Assuming that the acute angle BAC (Fig. 5), is one of the equal angles of an isosceles triangle, we construct the isosceles triangle ACB, which will include the given angle BAC as an angle at the base of the isosceles triangle. Circumscribe circumference ABC', from point C as a center and with AC as a radius. Bisect $\angle BCA$.

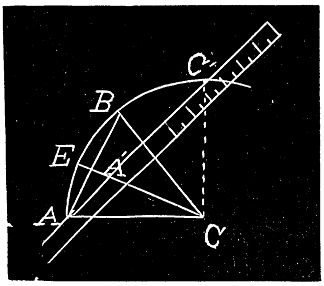


Fig. 5.

Then take the ruler and lay off on it a segment $A'C'$ equal to the radius AC. Bring the edge of the ruler to point A and draw line AC' in such a way that A' shall be on the bisector EC and C' on the circumference. The line AC' is the *trisectorial* line for $\angle BAC$.

PROOF. Join C' and C; triangle $A'C'C$ is isosceles, because $A'C'=C'C$ (by construction); triangle ACC' is also isosceles ($AC=CC'$).

Now, $\angle C'A'C=\angle C'AC+\angle A'CA$; $\angle C'CA'=\angle C'CB+\angle BCA'$. But $\angle BCA'=\angle A'CA$ (by construction). Hence $\angle C'CB=\angle C'AC$. But $\angle BCC'=2(\angle BAC')$ since the measure of $\angle BCC'$ is arc BC', and the measure of $\angle BAC'$ is one half of arc BC'.

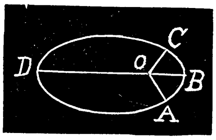
Hence, $\angle C'AC$ is also equal to $2(\angle BAC')$, and therefore $\angle BAC=3(\angle BAC')$, which makes the line AC' the *trisectorial* line.

A SIMPLE EXAMPLE OF A CENTRAL ORBIT WITH MORE THAN TWO APSIDAL DISTANCES.*

By DR. F. L. GRIFFIN, Williams College, Williamstown, Mass.

If a central force be a single-valued function of the distance, every orbit is symmetrical with respect to each apsidal line where the radius vector is a maximum or minimum, the apsidal angle is constant, and the orbit has not more than two apsidal distances.† These conclusions do not extend to forces which are multiple-valued. The purpose of this note is to call attention to an example of an orbit not having the properties mentioned above; it seems sufficiently simple to serve for class-room illustration.

Let $ABCD$ be an ellipse, and O be the point of intersection of two normals which make angles of 45° with the major axis. If this ellipse be described as a central orbit under a force directed to O, there will be four apsidal lines, OA, OB, OC, and OD, three of the apsidal distances being distinct. There are two distinct apsidal angles, $\frac{1}{4}\pi$ and $\frac{4}{3}\pi$. A similar state of affairs exists, if O be *any* point of the major axis, within the evolute of the ellipse (except the center).



The general case may be treated as follows. Let O be selected as the origin of co-ordinates; and let the center of the ellipse be at $(-d, 0)$, where $0<d<ae^2$. The equation of the ellipse is then:

$$\frac{(x+d)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1. \tag{1}$$

Or, using polar co-ordinates, and denoting by θ the longitude measured from OB, and by u the reciprocal of the radius vector, $[r]$, the equation becomes

$$(1-e^2)(\cos\theta+du)^2+\sin^2\theta=a^2(1-e^2)u^2, \tag{2}$$

or

$$\gamma\cos\theta=\alpha u\pm\sqrt{(\beta u^2+\gamma)}, \tag{3}$$

where $\gamma=e^2$, $\alpha=d(1-e^2)$, $\beta=(d^2-a^2e^2)(1-e^2)$, and the sign of the radical is to be determined.

At $\theta=0$, $u=\dfrac{1}{a-d}$, and, therefore,

$$(a-d)e^2=d(1-e^2)\pm\sqrt{[(d^2-a^2e^2)(1-e^2)+e^2(a-d)^2]},$$
$$\text{or } (a-d)e^2=d(1-e^2)\pm(ae^2-d).$$

*Read before the American Mathematical Society, April 27, 1907.
†E. J. Routh, *Dynamics of a Particle*, pp. 270-272.

Evidently the positive sign must be taken. At $\theta=\pi$, however, the reverse is true, for $u=\frac{1}{a+d}$, and

$$-(a+d)e^2=d(1-e)^2\pm(ae^2+d),$$

which requires the negative sign. Since $\cos\theta$ is obviously a continuous function of u in the ellipse, the sign of the radical in (3) can change only for a value of u which makes the radical vanish. There must be such a value taken somewhere in the orbit.

Now the apsidal values of u are given by $du/d\theta=0$.

Since $$-\gamma\sin\theta\frac{d\theta}{du}=\alpha\pm\frac{\beta u}{\sqrt{(\beta u^2+\gamma)}},\text{ or}$$

(4) $$\frac{du}{d\theta}=\frac{-\gamma\sin\theta\sqrt{(\beta u^2+\gamma)}}{\alpha\sqrt{(\beta u^2+\gamma)}\pm\beta u},$$

the values of u for which $du/d\theta=0$ are given by $\beta u^2+\gamma=0$ and $\sin\theta=0$. The values for which $\sin\theta$ vanishes are evidently $\frac{1}{a-d}$ and $\frac{1}{a+d}$. The values for which the radical vanishes are $u=\pm\sqrt{\frac{\gamma}{-\beta}}$, or $u_1=\frac{e}{\sqrt{[(a^2e^2-d^2)(1-e^2)]}}$, the negative value of u having no meaning. That this value of u (which is real, since $d<ae^2<ae$) is actually taken in the orbit, is seen from the value obtained for θ, corresponding to this value of u. Thus

(5) $$\cos\theta_1=\frac{\alpha u_1}{\gamma}=\frac{d\sqrt{(1-e^2)}}{e\sqrt{(a^2e^2-d^2)}},$$

which is real and less than unity; for $d^2<a^2e^4$, and hence $d^2(1-e^2)<e^2(a^2e^2-d^2)$.

There are, then, four apsidal lines, $\theta=0$, $\theta=\pm\theta_1$, $\theta=\pi$, as in the special case first mentioned, where $\theta_1=\frac{1}{2}\pi$. The apsidal angles are θ_1 and $\pi-\theta_1$, which are equal only for $\theta_1=\frac{1}{2}\pi$; *i. e.*, except for $d=0$, (when the center of force is at the center of the ellipse), there are two distinct apsidal angles. Evidently the sign of the radical in equation (3) must be positive from $\theta=-\theta_1$ to $\theta=+\theta_1$, and negative from $\theta=\theta_1$ to $\theta=2\pi-\theta_1$.

It remains to ascertain whether the radius vector is a maximum or minimum at each apse. It is easily seen that $u_1>\frac{1}{a-d}>\frac{1}{a+d}$. For, from $(d-ae^2)^2>0$, follows $e^2(a-d)^2>(a^2e^2-d^2)(1-e^2)$, or

$$u_1 = \frac{e}{\sqrt{[(a^2e^2-d^2)(1-e^2)]}} > \frac{1}{a-d}.$$

Since the radius vector can have maxima and minima only at $\theta=0$, $\theta=\pm\theta_1$, and $\theta=\pi$, and since r_1 $[=1/u_1]<(a-d)<a+d$, the radius vector is a maximum at $\theta=0$ and $\theta=\pi$, and is a minimum at $\theta=\pm\theta_1$.

Since the ellipse is asymmetrical with respect to the lines $\theta=\pm\theta_1$, it follows from the statement made at the outset that the force is a multiple-valued function of the distance. This conclusion may be verified by deriving the law of force. Thus,

$$f=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{h^2\gamma^2(\beta\gamma+a^2-\beta)u^2}{[\beta u\pm a\sqrt{(\beta u^2+\gamma)}]^3}, \tag{6}$$

where h is the constant of areas, and the positive and negative signs are to be given as in (3). Thus, for values of r between r_1 and $(a-d)$, the required force is a multiple-valued function.

The force has one striking peculiarity. For $u>u_1$, $\beta u^2+\gamma<0$, so that the force is imaginary. From this fact it easily follows that the force is such as not to permit a real orbit anywhere within a circle of radius r_1 about the center of force.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

NOTE ON PROBLEM 266. While in Washington, D. C., during the latter part of last August, I called on Mr. Theodore L. DeLand of the United States Treasury Department. While there, he called my attention to the usage among practical computers and actuaries in England regarding the finite series. He showed me a number of books in which infinite series were used and in which it was assumed that the series is determined by the terms that are given.

Thus, in the series $1+7+12+21$, it is assumed that the series has for its first differences, 5, 9, etc.; for the second differences, 4, 4, 4, etc. If only three terms were given, it would be assumed that the first differences are 5, 5, 5, etc. The series $1+3+7+17+\ldots$ of which the sum of n terms

were required becomes, with the understanding that its third differences are 2, 2, 2, etc., a definite series. I doubt, however, whether this convention adopted among many actuaries in England, is generally agreed to among mathematicians. Mr. DeLand based his solution of problem 266 on this assumption. Were this convention adopted, no ambiguity would arise in extending the series and finding its sum. ED. F.

GEOMETRY.

319. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

*Given the radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

Solution by the PROPOSER.

Let $AO=a$, $BP=b$, $CQ=c$, $AB=l$, $BC=m$, $CA=n$, $AD=q$, $BD=r$, $CD=p$, $\angle ADB=\theta$, $\angle BDC=\phi$, $\angle ADC=\psi$. Then $\cos\theta=\cos(\phi\pm\psi)$, and

$$\cos^2\theta+\cos^2\phi+\cos^2\psi-2\cos\theta\cos\phi\cos\psi=1.$$

$$\cos\theta=\frac{q^2+r^2-l^2}{2qr},\quad \cos\phi=\frac{r^2+p^2-m^2}{2rp},\quad \cos\psi=\frac{p^2+q^2-n^2}{2pq}.$$

Hence, $l^2(p^2-q^2)(p^2-r^2)+m^2(q^2-r^2)(q^2-p^2)+n^2(r^2-p^2)(r^2-q^2)$
$+l^2p^2(l^2-m^2-n^2)+m^2q^2(m^2-n^2-l^2)+n^2r^2(n^2-l^2-m^2)$
$+l^2m^2n^2=0 \ldots (1)$.

I. In (1), let $p=\pm(c-x)$, $q=\pm(a-x)$, $r=\pm(b-x)$.

*This problem, celebrated in the History of Mathematics, and also known as the "Tangency Problem," was first proposed and solved by Appolonius of Pergae, 200 B. C. Although this solution was lost for 1800 years, it was finally restored in 1600 A. D. by Vieta who, by reducing the original problem to a simpler form and thus solving simpler problems, gave an indirect solution.

The first direct solutions were furnished by Gaultier, 1813, and by Gergonne, 1814. The latter's method of solution is recorded in Carr's *Synopsis of Pure Mathematics*, page 224.

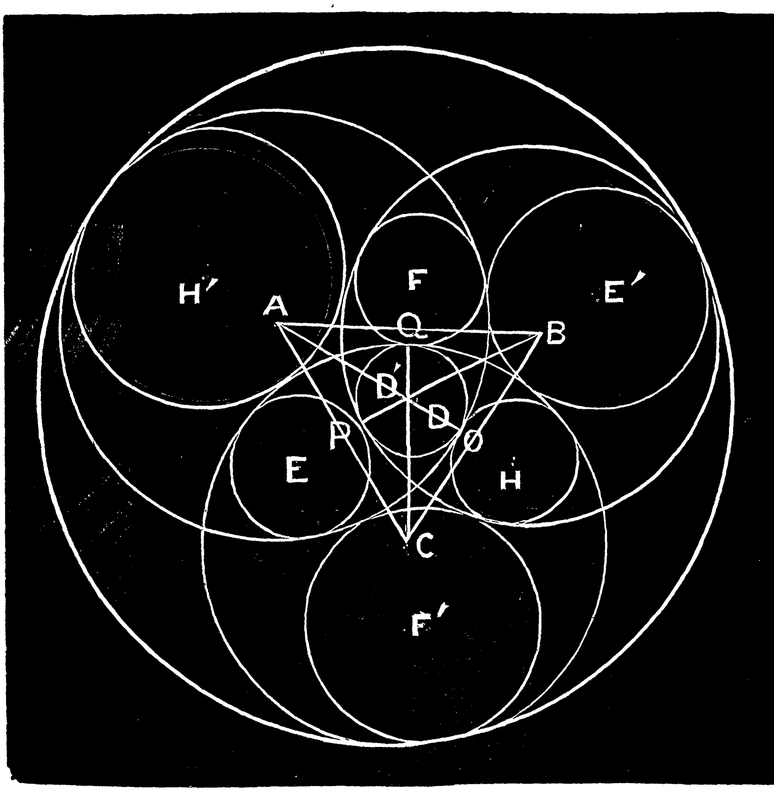
The first solution, though indirect, of its analogue in Solid Geometry, "To find a sphere touching four given spheres," was given by Fermat, 1665.

During the past century, numerous and varied geometrical solutions of this problem have appeared in many of the mathematical journals of Europe and America. The special problem, "Given the radii of three tangent circles to compute the radii of the two circles tangent to the three given circles," has been solved in various ways. From the very ingenious solution by Professor Enoch Beery Seitz, *School Visitor*, Vol. II, page 117, Mr. D. H. Davidson invented a method, *ibid*, Vol. VI, pages 80-84, of easily filling out a series of circles, beginning with any three given tangent circles.

The object of this solution is to complete the investigation by computing the radii of the eight circles touching any three circles, having given their radii and the distances apart of their centers. ZERR.

Two solutions of this problem are given in Vol. I, pages 220-222 of Leyborne's *Mathematical Questions*, and in Vol. IV, pages 259-275, are given Simson's, Vieta's, and Cauchy's solutions, together with a solution by Binet, and a trigonometric solution which appears to be by Leyborne himself, and a solution by Poncelet. A number of references and historical notes are there given. ED. F.

Then, $x^2[4l^2(c-a)(c-b)+4m^2(a-b)(a-c)+4n^2(b-a)(b-c)$
$-(m+n+l)(m+n-l)(m-n+l)(n+l-m)]$
$-2x[l^2(c-a)(c-b)(a+b+2c)+m^2(a-b)(a-c)(b+c+2a)$
$+n^2(b-a)(b-c)(a+c+2b)+cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$+bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-l^2-n^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0 \ldots (2).$



If, in (2), $l=a+b$, $m=b+c$, $n=a+c$, we get

$$x^2[4abc(a+b+c)-(ab+ac+bc)^2]-2xabc(ab+ac+bc)-a^2b^2c^2=0 \ldots (3).$$

From which, $x=-\dfrac{abc}{ab+ac+bc\pm 2\sqrt{[abc(a+b+c)]}} \ldots (4).$

We have thus found the radii of the circles having centers D and D', when the circles intersect, are tangent, or are non-tangent.

II. Let $p=c\pm x,\ q=a\pm x,\ r=b\mp x$.

Then, $x^2[4l^2(c-a)(c+b)+4m^2(a+b)(a-c)+4n^2(b+a)(b+c)$
$-(m+n+l)(m+n-l)(m-n+l)(n-m+l)]$
$\pm 2x[l^2(c-a)(c+b)(a-b+2c)+m^2(a+b)(a-c)(2a+c-b)$
$+n^2(b+a)(b+c)(a+c-2b)+cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$-bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(5).$

Equation (5) gives the radii of circles having the centers E and E' for intersection or non-tangency.

If, in (5), $l=a+b,\ m=b+c,\ n=a+c$, we have

$$x^2+2bx+b^2=0, \text{ or } x=-b\ldots(6).$$

III. Let $p=c\pm x,\ q=a\mp x,\ r=b\pm x$.

Then, $x^2[4l^2(c+a)(c-b)+4m^2(a+b)(a+c)+4n^2(b+a)(b-c)$
$-(m+n+l)(m+n-l)(m-n+l)(n-m+l)]$
$\pm 2x[l^2(c+a)(c-b)(b-a+2c)+m^2(a+b)(a+c)(b+c-2a)$
$+n^2(b+a)(b-c)(c-a+2b)+cl^2(l^2-m^2-n^2)-am^2(m^2-n^2-l^2)$
$+bn^2(n^2-l^2-m^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(7).$

Equation (7) gives the radii of the circles having the centers F, F' for intersection and non-tangency.

If, in (7), $l=a+b,\ m=b+c,\ n=a+c$, we have

$$x^2+2cx+c^2=0, \text{ or } x=-c\ldots(8).$$

IV. Let $p=c\mp x,\ q=a\pm x,\ r=b\pm x$.

Then, $x^2[4l^2(c+a)(c+b)+4m^2(a-b)(a+c)+4n^2(b-a)(b+c)$
$-(l+m+n)(l+m-n)(l-m+n)(m-l+n)]$
$\pm 2x[l^2(c+a)(c+b)(b+a-2c)+m^2(a-b)(a+c)(b-c+2a)$
$+n^2(b-a)(b+c)(a-c+2b)-cl^2(l^2-m^2-n^2)+am^2(m^2-n^2-l^2)$
$+bn^2(n^2-m^2-l^2)]+[c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$+(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)]=0\ldots(8).$

Equation (8) gives the radii of the circles having H and H' for centers both for intersection and non-tangency.

If, in (8), $l=a+b$, $m=b+c$, $n=a+c$, we have

$$x^2+2ax+a^2=0, \text{ or } x=-a \ldots (9).$$

Consider the following special case.

Let $a=12$, $b=10$, $c=8$, $l=15$, $m=11$, $n=13$, and put the equations for the four cases in the general form,

$$Ax^2-2Bx+C=0 \ldots (10),$$
$$\text{or } x=\frac{B\pm\sqrt{[B^2-AC]}}{A}.$$

Then, $C=(a^2b^2+c^2l^2)(l^2-m^2-n^2)+(b^2c^2+a^2m^2)(m^2-n^2-l^2)$
$+(a^2c^2+b^2n^2)(n^2-l^2-m^2)+c^4l^2+a^4m^2+b^4n^2+l^2m^2n^2$
$=-3099803$, the same for all cases.
$(l+m+n)(l+m-n)(l-m+n)(m-l+n)=77571$;
$cl^2(l^2-m^2-n^2)=-117000$; $am^2(m^2-n^2-l^2)=-396396$;
$bn^2(n^2-l^2-m^2)=-299130$.

Then, for Case I, $A=-69203$, $B=-730510$, and $x=2.3929$ or 18.7192.
Case II, $A=167917$, $B=-271610$, and $x=\pm2.6775$ or ±5.9125.
Case III, $A=129133$, $B=-346198$, and $x=\pm2.9041$ or ±8.2659.
Case IV, $A=241453$, $B=-112702$, and $x=\pm3.1465$ or ±4.0801.

As another example, let a, b, c, be the same as before, but let $l=30$, $m=20$, $n=25$. Then we find

$C=113822500$; $(l+m+n)(l-m+n)(l+m-n)(m-l+n)=984375$;
$cl^2(l^2-m^2-n^2)=-900000$; $am^2(m^2-n^2-l^2)=-5400000$;
$bn^2(n^2-m^2-l^2)=-4218750$.

Then, Case I, $A=-1010375$, $B=-10757950$, and $x=-4.3868$, or 25.6818.
Case II, $A=-112775$, $B=-2473250$, and $x=\pm16.6718$, or ±60.5335.
Case III, $A=-314375$, $B=-838750$, and $x=\pm16.5460$, or ±21.8820.
Case IV, $A=285625$, $B=-6898750$, $x=\pm10.5564$, and ±37.7500.

CALCULUS.

243. Proposed by R. D. CARMICHAEL, Anniston, Ala.

The usual method for the solution of a differential equation in the form (see Cohen, *Differential Equations*, p. 22)

$$x^ry^s(my\,dx+nx\,dy)+x^\rho y^\sigma(\mu y\,dx+\nu x\,dy)=0$$

fails when (1) $n=am$, (2) $\nu=a\mu$, (3) $s-\sigma\neq a(r-\rho)$. Find the solution when the relations (1) and (2) hold. (Note that the solution desired does not depend on (3).)

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

By making the substitution indicated, we get

$$mx^ry^s(ydx+axdy)+\mu x^\rho y^\sigma(ydx+axdy)=0.$$

$\therefore ydx+axdy=0$, and $mx^ry^s+\mu x^\rho y^\sigma=0$.

$\therefore \dfrac{dx}{x}+a\dfrac{dy}{y}=0$, and, therefore, $\log x+a\log y=\log C$, or $\log(xy^a)=\log C$ or $xy^a=C$, therefore, $x=C/y^a$.

$\therefore mC^r/y^{ar-s}+\mu C^\rho/y^{a\rho-\sigma}=0$, therefore, $y^{ar-s-a\rho+\sigma}=-\dfrac{m}{\mu}C^{r-\rho}$.

$\therefore y^{a(r-\rho)-(s-\sigma)}=-\dfrac{m}{\mu}C^{r-\rho}$, and, therefore, $y=(-\dfrac{m}{\mu}C^{r-\rho})^{1/[a(r-\rho)-(s-\sigma)]}$,

and $x=C(-\dfrac{m}{\mu}C^{r-\rho})^{a/[(s-\sigma)-a(r-\rho)]}$.

244. Proposed by G. B. M. ZERR, Ph. D., Professor of Mathematics in Central Manual Training School, Philadelphia, Pa.

Fine the volume common to the solids bounded by the surfaces $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$ and $x^{1/3}z^{2/3}=(a^{1/3}-x^{1/3})(x^{2/3}+y^{2/3})$.

Solution by the PROPOSER.

The limits of z are $z=\left[\dfrac{a^{\frac{1}{3}}-x^{\frac{1}{3}}}{x^{\frac{1}{3}}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})\right]^{\frac{3}{2}}$ to $z=(a^{\frac{2}{3}}-x^{\frac{2}{3}}-y^{\frac{2}{3}})^{\frac{3}{2}}$. Eliminating z from the equations, the limits of y are $y'=[x^{\frac{1}{3}}(a^{\frac{1}{3}}-x^{\frac{1}{3}})]^{\frac{3}{2}}$ and 0; of x, 0 and a.

$$\therefore V=4\int_0^a\int_0^{y'}\left[(a^{\frac{2}{3}}-x^{\frac{2}{3}}-y^{\frac{2}{3}})^{\frac{3}{2}}-\left(\frac{a^{\frac{1}{3}}-x^{\frac{1}{3}}}{x^{\frac{1}{3}}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})\right)^{\frac{3}{2}}\right]dx\,dy.$$

Let $x=u^3$, $y=v^3$, $a=b^3$. Then we get

$$V=36\int_0^b\int_0^{\sqrt{[u(b-u)]}}u^2v^2\{(b^2-u^2-v^2)^{\frac{3}{2}}-[\frac{b-u}{u}(u^2+v^2)]^{\frac{3}{2}}\}du\,dv$$

$$=\tfrac{9}{4}\int_0^b[u^2(b^2-u^2)^3\sin^{-1}\sqrt{\frac{u}{b+u}}+u^5(bu-u^2)^{\frac{3}{2}}\log\left(\frac{\sqrt{(b-u)}+\sqrt{b}}{\sqrt{u}}\right)$$

$$+bu^2(b-u)^2(bu-b^2-u^2)\sqrt{(bu)}]du.$$

In the first term let $u=b\tan^2\theta$, in the second and third terms let $u=b\sin^2\theta$. Then

$$V=\tfrac{9}{2}b^9\int_0^{\frac{1}{4}\pi}\theta\tan^5\theta(1-\tan^2\theta)^3\sec^8\theta\,d\theta+\tfrac{9}{2}b^9\int_0^{\frac{1}{2}\pi}\sin^{14}\theta\cos^4\theta\log\frac{1+\cos\theta}{\sin\theta}d\theta$$

$$+\tfrac{9}{2}b^9\int_0^{\frac{1}{2}\pi}\sin^6\theta\cos^5\theta(\cos^2\theta\sin^2\theta-1)\,d\theta$$

$$=\tfrac{2}{35}\pi\, a^3-\frac{63074663899a^3}{390329139200}+\frac{1287a^3}{131072}\int_0^{\frac{1}{2}\pi}\log\cot\tfrac{1}{2}\theta\,d\theta$$

$$=\tfrac{2}{35}\pi\, a^3-\frac{63074663899a^3}{390329139200}+\frac{2389\,\pi^2a^3}{1310720}\text{ nearly.}$$

MECHANICS.

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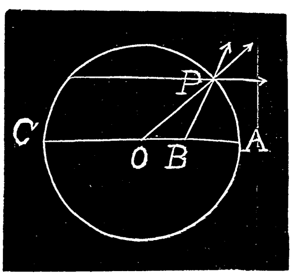
203. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A train weighing $T(=80)$ tons runs first eastward and then westward in latitude $\lambda(=40^\circ)$ at a velocity $v(=45)$ miles an hour. Find the difference between the pressures on the ground in the two cases.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

In the figure, let P be the point of the train on the λth ($=40$th) parallel, O the center of the earth, PB the normal at P, $\angle POA=\theta$, $\angle PBA=\lambda$, $OA=a=20923536$ feet, $OP=r$, $CP=2\rho$, $e=$the earth's ellipticity, $e^2=.006920928$, $F=$centrifugal force in direction OP, and $f=$centrifugal force in direction CP. Then $\rho=r\cos\theta=a\cos\lambda/\sqrt{(1-e^2\sin^2\lambda]}=16051229$ feet.



$V_P=\dfrac{2\pi\rho}{t}$, where $t=1$ day, $=\dfrac{2\pi\rho'}{86400}=$ 1167.28 feet per second, the velocity of P due to the earth's rotation.

V_P-V_R, where $V_R=$train's velocity in feet per second, $=1167.28$ feet-66 feet$=1101.28$ feet, train's velocity in space going west.

$V_P+V_R=1167.28$ feet$+66$ feet$=1233.28$ feet, train's velocity in space going east.

$f=TV_1^2/g\rho=TV_1^2/(gr\cos\theta)$, $F=f\cos\lambda=TV_1^2\sqrt{(1-e^2\sin^2\lambda)}/ag$, and $g=G(1+\frac{1}{4}e^2\sin^2\lambda)$, where $G=32.2015235$, gravity at the equator.

$\therefore g=32.2245411$ feet per second; $\therefore F=0.000000119V_1^2$ tons.

$F_W=0.1443253$ tons, going west; $F_E=0.15209796$ tons, going east.

Difference$=0.00777266$ tons$=15.545$ pounds. (See Vol. VI, No. 11, page 282, and Vol. IX, No. 2, page 32.)

Also solved by G. W. Greenwood and J. Scheffer. These gentlemen omitted the earth's ellipticity in their solutions and consequently their result differs from that of Dr. Zerr.

204. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A set of particles have coplanar motion due to mutual attractions. Each particle is now affected with a velocity V parallel to a fixed direction. How will this affect the angular momentum of the set about their centroid?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The forces that produce the velocities would produce a velocity on the centroid in the same direction equal to the algebraic sum of all the velocities. As these forces (external) produce no moment about the centroid, or, in other words, the sum of all the momenta about the centroid is zero, the angular momenta of all the particles about the centroid is constant. (See Routh's *Dynamics of a Particle*, Art. 260, page 159.)

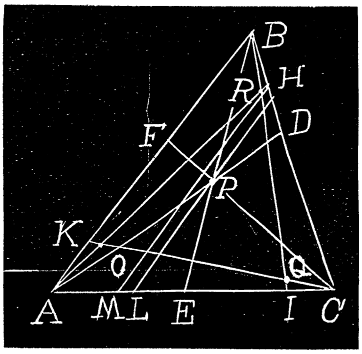
AVERAGE AND PROBABILITY.

189. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

(a) Lines are drawn from the vertices of a triangle through a random point within it. Find the average area of the triangle formed by joining the points of intersection of these lines with the opposite sides. (b) Lines are drawn from the vertices to points taken at random in the opposite sides of a triangle. Find the average area of the triangle formed by the intersections of these lines.

Solution by the PROPOSER.

(a) Let P be the random point in the given triangle, sides a, b, c. Draw AD, BE, CF, DE, EF, FD. Also draw PL parallel to AB. Let $AL=u$, $PL=v$.



Co-ordinates of E are $\left(\dfrac{cu}{c-v}, 0\right)$; of F, $\left(0, \dfrac{bv}{b-u}\right)$; of D, $\left(\dfrac{bcu}{cu+bv}, \dfrac{bcv}{cu+bv}\right)$.

Area $DEF=\dfrac{bcuv(bc-bv-cu)\sin A}{(c-v)(b-u)(cu+bv)}$

$=2\triangle G$, where $\triangle$ =area of ABC.

Limits of u are 0 and b; of v, 0 and $\dfrac{c}{b}(b-u)=v_1$. Let A=average area.

$$\therefore A=\frac{2\triangle\int_0^b\int_0^{v_1}G\,du\,dv}{\int_0^b\int_0^{v_1}du\,dv}=\frac{4\triangle}{bc}\int_0^b\int_0^{v_1}G\,du\,dv$$

$$=\frac{4\triangle}{bc}\int_0^b\left(\frac{cu}{b}+\frac{2cu^2}{b^2-u^2}\log\frac{u}{b}\right)du=4\triangle\int_0^1\left(x+\frac{2x^2}{1-x^2}\log x\right)dx,\text{ where } u=bx.$$

$\therefore A=\triangle(10-\pi^2)$.

(b) Let I, H, K be the random points in AC, BC, AB, respectively. Draw HM parallel to AB. Let $AI=u$, $AM=z$, $AK=v$. Then $MH=(c/b)(b-z)$.

Equation to AH is $y=\dfrac{c(b-z)x}{bz}$... (1).

Equation to BI is $y=c(u-x)/u$... (2).

Equation to CK is $y=v(b-x)/b$... (3).

(1) and (2) intersect in $x_1=\dfrac{buz}{bu-uz+bz}$, $y_1=\dfrac{cu(b-z)}{bu-uz+bz}$.

(1) and (3) intersect in $x_2=\dfrac{bvz}{bc-cz+vz}$, $y_2=\dfrac{cv(b-z)}{bc-cz+vz}$.

(2) and (3) intersect in $x_3=\dfrac{bu(c-v)}{bc-uv}$, $y_3=\dfrac{cv(b-u)}{bc-uv}$.

Area $OQR=\frac{1}{2}bc\sin A\left(\dfrac{[u(vz+bc-cz)-v(bu-uz+bz)]^2}{(vz+bc-cz)(bu-uz+bz)(bc-uv)}\right)=\triangle G$.

The limits of u are 0 and b; of z, 0 and b; of v, 0 and c.

$$\therefore A=\frac{\triangle\int_0^b\int_0^b\int_0^c G\,du\,dv\,dz}{\int_0^b\int_0^b\int_0^c du\,dv\,dz}=\frac{\triangle}{b^2c}\int_0^b\int_0^c\int_0^b G\,du\,dv\,dz$$

$$=\frac{\triangle}{bc}\int_0^b\int_0^c\left(\frac{u^2(v-c)}{(b-u)(bc-uv)}-\frac{u^2}{(b-u)^2}\log\frac{u}{b}-\frac{2uv}{bc-uv}\right)du\,dv$$

$$+\frac{\triangle}{bc}\int_0^c\int_0^b\left(\frac{v^2(u-b)}{(c-v)(bc-uv)}-\frac{v^2}{(c-v)^2}\log\frac{v}{c}\right)dv\,du$$

$$=\frac{2\triangle}{b}\int_0^b\left[1-\frac{b}{u}\log\left(\frac{b-u}{u}\right)-\frac{u}{b-u}-\log\frac{b-u}{b}-\frac{u^2}{(b-u)^2}\log\frac{u}{b}\right]du.$$

Let $bx=b-u$.

$$\therefore A=2\triangle\int_0^1\left[1-\frac{1}{1-x}\log\left(\frac{x}{1-x}\right)-\frac{1-x}{x}-\log x-\frac{(1-x)^2}{x^2}\log(1-x)\right]dx$$

$$=\triangle(10-\pi^2).$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

291. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

An empty water tank has two inflow pipes A, B, which begin to flow at the same moment. When B, the smaller pipe, has discharged s gallons, and the tank is $1/n$ filled, water from both pipes is turned off. After A, B, have been idle, each as many hours as would suffice it to perform $1/m$ the work done previously by the other pipe, the flow, which is of a uniform rate, is resumed and continued till the tank is filled; B during the second working period has discharged t gallons. (1) What is the capacity of the tank? (2) What would be the capacity if B were an outflow pipe?

292. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Find the sum of the series $1^2+5^2+14^2+30^2+\ldots+[\frac{1}{6}n(n+1)(2n+1)]^2$.

GEOMETRY.

324. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College, Williamstown, Mass.

Find all plane curves such that the normal lengths intercepted by the co-ordinate axes are in a constant ratio for all points.

325. Proposed by A. H. HOLMES, Brunswick, Maine.

An aeronaut, describing the earth's appearance from a certain height, said it seemed like an immense bowl with the horizon for its rim. (1) At what height would the apparent deepness of the "bowl" be the greatest? (2) To what height would the earth's surface again appear flat?

CALCULUS.

247. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Integrate, $x\frac{\partial^2 y}{\partial x^2}+2\frac{\partial y}{\partial x}-xy=0$.

248. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Evaluate $\int_0^{\frac{1}{2}\pi}\sin nx\cot x dx$, where n is a positive integer.

249. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Ike, running with constant velocity v, is trying to catch Jim, running with constant velocity V, $(V>v)$, by keeping Jim dead ahead of him. Find their paths.

MECHANICS.

208. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

Hanging at rest over a smooth pulley are two equal scale pans of the same mass. Two equal particles, the one inelastic and the other elastic, are simultaneously dropped from the same height one into each scale pan. Show that each impact after the first must occur when the pans have returned to the *status quo ante,* and find the total space described by either pan before motion ceases.

209. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, Eng.

Two particles are projected along planes at angles α and $\frac{1}{2}\pi-\alpha$ to the horizon, the horizontal lines on the two planes being inclined at an angle ϕ. The initial relative velocity is parallel to a certain plane. Show the relative path is a parabola, and find the inclination of its axis to the vertical.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

149. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

Prove that every prime of the form $4n+1$ may be expressed as the sum of two parts r and s such that r^2+rs+s^2+1 is divisible by the prime.

150. Proposed by H. S. VANDIVER, Bala, Pa.

Show that for all positive integral values of n except unity, $(2n)!$ is less than $[n(n+1)]^n$. Direct proof preferred. [Unsolved problem in *Educational Times.*]

AVERAGE AND PROBABILITY.

192. Proposed by REV. R. D. CARMICHAEL, Anniston, Ala.

A point is taken at random in a square whose side is $2a$. With this point as center and radius$=a$ a circumference is described. What is the mean area of that part of the circle which lies within the square?

BOOKS.

A Treatise on the Integral Calculus Founded on the Method of Rates. By William Woolsey Johnson, Professor of Mathematics at the United States Naval Academy, Annapolis, Md. Small 8vo. Cloth, xiv+440 pages, 71 figures. Price, $3.00. New York: John Wiley and Sons.

This volume is an enlargement and an extension of the author's *Elementary Treatise on the Integral Calculus,* a revised edition of which appeared in 1898, and forms a companion volume to his *Treatise on the Differential Calculus,* published in 1904.

The enlargement consists in a fuller treatment of formulae of reduction and of double and triple integrals and the extension, comprising about half of the volume, consists of a chapter (IV) on Mean Values and Probabilities; one (V) on Definite Integrals, including the Eulerian Integrals, Fourier's Series, and Chapter VI on Functions of a Complex Variable. Each section is followed by a large collection of problems with their answers.

This book, together with its companion volume, forms an admirable course for students of the Calculus. F.

Geometric Exercises for Algebraic Solution. Second Year Mathematics for Secondary Schools. By George William Myers, Professor of the Teaching of Mathematics and Astronomy, College of Education of The University of Chicago, and William R. Wicks, Ernest A. Wreidt, and Ernest R. Breslech, Instructors in Mathematics in the University High School of The University of Chicago. 8vo. Cloth, ix+71 pages. Price, 75 cents. Chicago: The University of Chicago Press.

The authors of this little volume felt the imperative need of enabling mathematical pupils, during the second year, at least to hold the algebraic ground and fill the gap between the first year algebra and the second year when it is usually dropped to take up geometry, and to this end they prepared this book of exercises. The exercises are so chosen as to cover algebra to and through quadratics, and they will be found exceedingly useful to teachers of either algebra or geometry. F.

First Year Mathematics for Secondary Schools. By George William Myers, Professor of the Teaching of Mathematics and Astronomy, College of Education of the University of Chicago, and William R. Wicks, Harris F. MacMeish, Ernest R. Breslich, Ernest A. Wreidt, Instructors in the University High School of The University of Chicago. 8vo. Cloth, xv+189 pages. Price, $1.00. Chicago: The University of Chicago Press.

First Year Mathematics is a stage of study of the practical educational problem of inducting beginners in secondary school mathematics into the diversified field of this science, through the agency of a body of unified mathematical material, drawn from algebra, arithmetic, geometry, and from the rudiments of quantitative science. The backbone of the year's work is algebra, into which the material drawn from kindred fields articulates. The book is not a finished text but a distinct step toward a text that shall be adapted to first-year maturity, and in conformity with the modern trend of educational thought.

A Text-Book in Physics for Secondary Schools. By William N. Mumper, Ph. D., Professor of Physics in the State Normal and Model Schools of Trenton, New Jersey. 8vo. Cloth, 411 pages. Price, $1.20. New York and Chicago: American Book Co.

This book takes up the study of physical laws where the pupil's knowledge respecting them begins; that is, it begins with the level of the pupil's knowledge. It treats the subjects in a systematic and concise manner. The illustrations are very good and the book is gotten up in splendid form. F.

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NOTE ON THE DEFINITIONS OF A VARIABLE.

By DR. G. A. MILLER.

Some mathematical definitions which present no logical difficulties seem at first so unnatural as to cause considerable comment. Among these are the definitions for 'variable number' and for the 'symbol which is a variable'. The natural numbers furnish examples of ideal invariants. Not only are they the same in every country but we think of them as the same throughout the universe. They are certainly included in Sir Oliver Lodge's interesting observation, "Real living arithmetic is the same in any country; and the most important of all is that which must necessarily be the same on any planet."*

Even when the concept of numbers has been extended so as to include all real and complex numbers we are in the habit of thinking of them as constants and finite.† One of the important uses of numbers is to express a ratio between two quantities of the same kind. One of these quantities may be supposed fixed while the other varies. If we represent this ratio by r it is convenient to speak of r as a variable. No mathematician would hesitate to do this but some would not like to call r a variable number while others do not see any objections to such a usage.

The concept of variable number is analogous to that of variable point. Just as we conceive of some curves as generated by a moving point so we may conceive of the numbers represented by the points of the curve as generated by a number whose elements are the coordinates of the point. In fact, the term point is frequently defined as an aggregate of numbers. The term variable point is so convenient that it is commonly used in all civilized countries,‡ and some authors who define a variable as a symbol to which we may assign successively different numbers do not hesitate to speak of a variable point.§

**Easy Mathematics, Chiefly Arithmetic;* By Sir Oliver Lodge, 1905, p. 64.

†In der Tat haben die Worte unendliche Zahl keinen Sinn, da die Zahl an und feur sich wesentlich endlich ist. Cesaro-Kowalewski, *Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung*, 1904, p. 183.

‡Cf. G. Peano, *Applicazioni Geometriche de Calcolo Infinitesimale*, 1887, p. 146; Picard et Simart, *Theorie des Fonctions algebriques de deux variables independentes*, 1906, p. 21.

§Weber and Wellstein, *Encyklopaedie der Elementar-Mathematik*, Vol. 2, 1905, page 477.

Before the days of Weierstrass writers generally defined the real variable x as a quantity or magnitude which is not constant. All are familiar with such variable magnitudes, for instance, the height of a growing tree. Hence such a definition may at first appear to be a very natural one and it is still employed by a good many writers. From a logical standpoint it seems to present serious difficulties* since the variable x is subjected to the ordinary laws of operation without any inquiry as to the meaning of these operations when applied to the different possible quantities. When the movement towards arithmetizing mathematics began it seemed best to many to replace the term quantity or magnitude by the more definite term number.

While the variable usually stands for any one of a series of numbers yet it is not always desirable to think of it in this way. For instance, in the theory of algebraic numbers the symbols for the variables stand simply for things with which we may operate and have no independent meaning.† In view of this fact many modern writers prefer to define the variable as a symbol to which we may assign various meanings according to the subject under consideration. For instance, in the infinitesimal calculus and function theory the real variable may be defined as a symbol to which we may assign successively different numbers.‡ Whether these numbers constitute all the possible numbers in a given interval or only some of them, such as the integers or multiples of a given number, does not affect the definition.

If it is granted that the real variable may be defined in analytic geometry and calculus as a symbol to which different fixed numbers may be assigned successively, it does not follow directly that this is the most suitable definition for the majority of the students who pursue these courses. The initial concept of symbol carries with it the idea of constancy almost as fully as the initial concept of number does and it is doubtful whether the term variable number offers greater difficulty to the student than the expression the symbol x is a variable. The former is used by such careful writers as Peano, Genocchi-Peano, *Differentialrechnung und Grundzuge der Integralrechnung*, translated into German by Bohlmann and Schepp, 1899, p. 3; and Jordan, *Cours d'Analyse*, Vol. 1, 1893, p. 172.§

It is not to be inferred that those who define the variable as a symbol necessarily imply that the term variable number would be objectionable. In many cases the desire to avoid expressions which may present difficulties *per se* in order to fix the attention upon more important things has doubtless been a more potent motive. The expression the symbol x is a variable is somewhat shorter than the symbol x represents a variable number, and it

*Cf. Pierpont, *Bulletin of the American Mathematical Society*, Vol. 5, 1899, p. 396.

†In particular cases it is desirable to replace these variables successively by fixed numbers. Cf. Weber, *Lehrbuch der Algebra*, Vol. 2, 1899, p. 568.

‡This definition is adopted by a large number of authors; Cf. Pringsheim, *Encyklopaedie der Mathematischen Wissenschaften*, Vol. 2, p. 8.

§In his *Lecons sur les fonctions de variables reelles*, 1905, Borel frequently speaks of nombre fixe, which implies the existence of others; *e. g.* p. 39.

is more convenient to speak of the variable x than the variable number represented by x. Hence the symbol definition of a variable has a slight advantage with respect to brevity and it makes it easier for those whose desire to avoid the term variable number.

The main object of the preceding remarks is to call attention to the principal elements involved in the different definitions of the term variable, and to give some reasons for the existence of these differences. The teacher is frequently called on to justify his views on points where different usages exist and it is desirable to be able to give references as well as reasons even if the difference appears somewhat trivial. If all numbers were essentially constants I presume most of us would agree with the foot-note quoted from Cesaro to the effect that there is no such thing as an infinite number, for we cannot conceive of any constant number which has such a distinctive character; but it is not so difficult to conceive of a variable which increases without limit and it is convenient to express this concept by saying that the variable is infinite. The ideas of infinite number and variable number are therefore closely related. The existence of the number zero, on the contrary, does not necessarily involve the concept of variability.

NOTE ON A FORMULA FOR THE SUMMATION OF CERTAIN POWER SERIES.

By HENRY KEMMERLING, Scranton, Pa.

Weierstrass's factor-theorem* is

$$fx=e^{Gx}\prod_{n=1}^{\infty}\left(1-\frac{x}{a_n}\right)e^{x/a_n+\frac{1}{2}(x/a_n)^2+[1/(n-1)]\,(x/a_n)^{n-1}},$$

where fx has zeros at the points a_1, a_2, a_3, ...

Taking logarithms and differentiating,

$$\frac{f'x}{fx}=G'x+\sum_{n=1}^{\infty}\left(\frac{1}{x-a_n}+\frac{1}{a_n}+\frac{x}{{a_n}^2}+\ldots\frac{x^{n-2}}{{a_n}^{n-1}}\right).$$

Then if $\left(\dfrac{f'x}{fx}-G'x\right)$ is Px, we have

*Harkness and Morley's *Introduction to Analytic Functions*, p. 199

$$Px=\sum_{n=1}^{\infty}\left(\frac{1}{x-a_n}+\frac{1}{a_n}+\frac{x}{a_n^{\,2}}=\ldots\frac{x^{n-2}}{a_n^{\,n-1}}\right)=-\sum_{n=1}^{\infty}\left(\frac{x^{n-1}}{a_n^{\,n}}+\frac{x^n}{a_n^{\,n+1}}+\ldots\right). \tag{1}$$

Maclauren's formula is

$$Px=P0+P'0\frac{x}{1}+\ldots\, P^n0\frac{x^n}{n!}+\ldots \tag{2}$$

Comparing the general terms of (1) and (2), we have

$$P^n0\frac{x^n}{n!}=-\sum_{n=1}^{\infty}\frac{x^n}{a_n^{\,n-1}},\ \text{or}\ P^n0=-n!\sum_{n=1}^{\infty}\frac{1}{a_n^{\,n+1}}. \tag{3}$$

As an application of (3) take $fx=x^3-6x^2+11x-6$, whose zeros a_1, a_2, a_3, are 1, 2, 3.

$f'x=3x^2-12x+11.$

$$Px=\frac{f'x}{fx}=\frac{11-12x+3x^2}{-6+11x-6x^2+x^3},$$

which by division,

$$=-\tfrac{11}{6}-\tfrac{49}{36}x-\tfrac{251}{216}x^2-\ldots$$

Using (3), and reducing, we have,

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}=\tfrac{11}{6},$$

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}=\tfrac{49}{36},$$

$$\frac{1}{a_1^3}+\frac{1}{a_2^3}+\frac{1}{a_3^3}=\tfrac{251}{216},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Again let $fx=1-\sin\frac{\pi x}{2}$, which has *two* zeros at each of the points $+1, -3, +5, -7, \ldots$

$$f'x=-\frac{\pi}{2}\cos\frac{\pi x}{2},$$

$$Px=\frac{f'x}{fx}=\frac{-(\pi/2)\cos(\pi x/2)}{1-\sin(\pi x/2)},$$

$$P'x=\frac{-(\pi^2/4)}{1-\sin(\pi x/2)},$$

$$P''x=\frac{-(\pi^3/8)\ \cos(\pi x/2)}{[1-\sin(\pi x/2)]^2},$$

$$P'''x=\frac{(\pi^4/16)\sin(\pi x/2)[1-\sin(\pi x/2)]-(\pi^4/8)\ \cos^2(\pi x/2)}{[1-\sin(\pi x/2)]^3}.$$

$P0=-\pi/2,\quad P'0=-\pi^2/4,\quad P''0=-\pi^3/8,\quad P'''0=-\pi^4/8.$

Applying (3),

$$-\frac{\pi}{2}=-2\left(\frac{1}{1}+\frac{1}{-3}+\frac{1}{5}+\frac{1}{-7}+\dots\right)$$

$$-\frac{\pi^2}{4}=-2\left(\frac{1}{1^2}+\frac{1}{(-3)^2}+\frac{1}{5^2}+\frac{1}{(-7)^2}+\dots\right)$$

$$-\frac{\pi^3}{8}=-4\left(\frac{1}{1^3}+\frac{1}{(-3)^3}+\frac{1}{5^3}+\frac{1}{(-7)^3}+\dots\right)$$

$$-\frac{\pi^4}{8}=-12\left(\frac{1}{1^4}+\frac{1}{(-3)^4}+\frac{1}{5^4}+\frac{1}{(-7)^4}+\dots\right)$$

Reducing above we get the familiar series,

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots$$

$$\frac{\pi^3}{32}=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots$$

$$\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\dots$$

. . . .

NOTES ON THE RADICAL AXIS AND THE QUADRILATERAL.

By WILLIAM GALLATLY, Boscombe, England.

I. The Radical Axis.

1. If d be the distance of a point P from the center of a circle X of radius x, then $\pm(d^2-x^2)$ is called the power of P with respect to X. ABC being the triangle of reference, let ρ_1^2, ρ_2^2, ρ_3^2 be the powers of A, B, C, and let the circle X have the equation

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+k(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma)=0.$$

Now if $S=0$ be any conic, and if S' be the value of S when the coordinates of P, Cartesian or trilinear, are substituted for the general coordinates, it is known that S' is proportional to the power of P. Hence, for the point A, $\lambda\rho_1^2=k.2\triangle.l.2\triangle/a$; for A is $(2\triangle/a,\ 0,\ 0)$. Hence $kl=\lambda/4\triangle^2.a\rho_1^2$.

The radical axis of the circles X and ABC is thus

$$a\rho_1^2.\alpha+b\rho_2^2.\beta+c\rho_3^2.\gamma=0.$$

And the circle X takes the form

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+\frac{\lambda}{4\triangle^2}.(a\alpha+b\beta+c\gamma)(a\rho_1^2.\alpha+b\rho_2^2.\beta+c\rho_3^2.\gamma)=0.$$

2. We now proceed to find the radical axis of the circle ABC, and (1) the in-circle; (2) the nine point circle; (3) the polar circle; (4) the circle on Hb (H being the orthocenter).

For (1), $\rho_1^2=(s-a)^2$; so that the radical axis is $a(s-a)^2.\alpha+....=0$.

For (2), N being the nine point center,

$$\rho_1^2=AN^2-Nd^2=Ad^2+2.Ad.dN.\cos(B-C)$$
$$=R^2\cos A[\cos A+\cos(B-C)]=2R^2\cos A\ \sin B\ \sin C.$$

Hence $a\rho_1^2$ varies as $\cos A$. The radical axis is $\cos A.\alpha+....=0$.

For (3). The polar circle has center H, and $x^2=-4R^2\cos A\ \cos B\cos C$. Therefore, $\rho_1^2=AH^2-k^2=4R^2\cos^2A+4R^2\cos A\ \cos B\ \cos C=4R^2\ \cos A\ \sin B\times\sin C$. Hence $a\rho_1^2$ varies as $\cos A$, and the radical axis is $\cos A.\alpha+....=0$.

For (4). Let ω be the mid-point of GH.

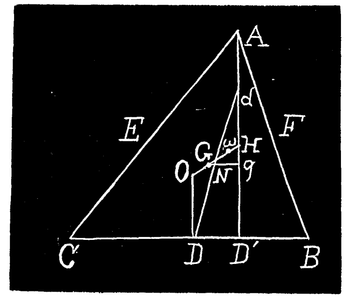
$$2\rho_1^2=2(A\omega^2-\omega H^2)=2(A\omega^2+\omega H^2)-Hb^2=Ab^2+AH^2-HG^2$$
$$=2AG.AH.\cos GAH=2AH.Ag=2.AH.\tfrac{2}{3}.AD'$$
$$=4R\ \cos A.\tfrac{2}{3}.2R\ \sin B\ \sin C.$$

Hence $a\rho_1^2$ varies as $\cos A$, and the radical axis is $\cos A.\alpha+\ldots.=0$.

3. To determine the radical axis of the nine point circle, and (1) circum-circle, (2) in-circle.

It is now convenient to take the mid-point triangle DEF *as triangle of reference.*

If $(\alpha'\beta'\gamma')$ and $(\alpha\beta\gamma)$ be the old and new coordinates of a point, then $\alpha'+\alpha=\frac{1}{2}.AD'=2\triangle/a$; all symbols now referring to DEF, so that $a\alpha'=b\beta+c\gamma$; and $l\alpha+m\beta+n\gamma=0$ becomes



$$a\alpha\left(\frac{m}{b}+\frac{n}{c}\right)+\ldots.=0.$$

Hence $\cos A.\alpha+\ldots.=0$, becomes $a^3\alpha+\ldots.=0$.

For (1), $x=2R$; $d=OD=2R\cos A$.

Hence $\rho_1^2=4R^2(1-\cos^2 A)$. Therefore $a\rho_1^2$ varies as a^3, so that the radical axis is $a^3.\alpha+\ldots.=0$, agreeing with Art. 2 (1).

For (2), $\rho_1^2=(b-c)^2$; so that the radical axis is $a(b-c)^2.\alpha+b(c-a)^2.\beta+c(a-b)^2.\gamma=0$. But this is the tangent to $a/\alpha+\ldots.=0$ (the nine point circle) at

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}\right).$$

Hence (Feuerbach's Theorem), the nine point and in-circles touch; the point of contact being

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}\right).$$

II. The Quadrilateral.

1. Let the triangle ABC be cut by the straight line

$$\frac{a\alpha}{l}+\frac{b\beta}{m}+\frac{c\gamma}{n}=0$$

in the points A', B', C'. Bisect AA', BB', CC' in A_1, B_1, C_1.

For A_1, $\dfrac{b\beta}{m}+\dfrac{c\gamma}{n}=0$, and $b\beta+c\gamma=a\alpha$.

$\therefore \dfrac{b\beta}{m}=\dfrac{c\gamma}{-n}=\dfrac{a\alpha}{m-n}=\dfrac{\triangle}{m-n}$, with similar expressions for B'. Hence

$$A'B' \text{ is } (m+n-l)a\alpha+(n+l-m)b\beta+(l+m-n)c\gamma=0,$$

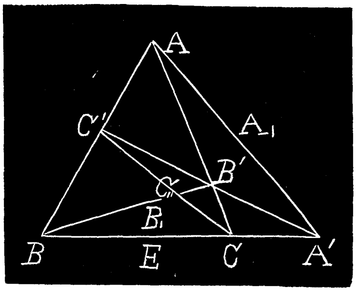
the symmetry of the equation showing that C' is also on this line.

2. If $BCB'C'$ is circumscriptible to a circle, then the mid-point line passes through the center. The in-circle of ABC is $\surd[a(s-a).\alpha+....=0$.

If $A'B'C'$ is a tangent, then $\frac{a(s-a)}{a/l}+....=0$, or $(m+n-l)a+....=0$.

Hence the centre (1, 1, 1) lies on the mid-point line.

3. The orthocenters of the four triangles ABC, $AB'C'$, $BC'A'$, $CA'B'$, are collinear.



The perpendiculars from B' on AB, and C' on AC, are found to be

$$na.\cos A.\alpha+(lc-na\cos B)\beta+lc\cos A.\gamma=0.$$
$$ma.\cos A.\alpha+lb\cos A.\beta+(lb-ma\cos c)\gamma=0.$$

By subtraction we obtain the symmetrical equation

$$(m-n)\cos A.\alpha+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

which proves the proposition.

4. To find the equations of the circles $AB'C'$, $BC'A'$, $CA'B'$. Let the equation of $AB'C'$ be

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=k(a\alpha+b\beta+c\gamma).(m'\beta+n'\gamma) \quad (l' \text{ obviously zero}).$$

At B', $\frac{a\alpha}{l}+\frac{c\gamma}{n}=0$; $\beta=0$. Hence, $\frac{a\alpha}{-l}=\frac{c\gamma}{n}=\frac{2\triangle}{n-l}$; and $b\gamma\alpha=k.2\triangle.n'\gamma$.

Hence, $kn'=\frac{b}{a}.\frac{-l}{n-l}$; so $km'=\frac{c}{a}.\frac{l}{l-m}$.

Hence, the equation of $AB'C'$ is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=\frac{l}{a}(a\alpha+b\beta+c\gamma)\left(\frac{c}{l-m}.\beta-\frac{b}{n-l}.\gamma\right);$$

so that, if the circle $AB'C'$ cuts the circle ABC in D, the equation to AD is

$$\frac{c}{l-m}.\beta-\frac{b}{n-l}.\gamma=0, \text{ or } \beta\div\frac{b}{n-l}=\gamma\div\frac{c}{l-m}.$$

It follows that the circles $AB'C'$, $BC'A'$, $CA'B$ cut the circle ABC in the same point D, whose coordinates are $\left(\frac{a}{m-n}, \frac{b}{n-l}, \frac{c}{l-m}\right)$.

5. The condition that $BCB'C'$ may be cyclic, or that $B'C'$ may be anti-parallel to BC. In this case it can be at once proved geometrically that D lies on AA' or $\frac{b\beta}{m}+\frac{c\gamma}{n}=0$. Hence the required condition is

$$\frac{b^2}{m(m-n)}+\frac{c^2}{n(l-m)}=0.$$

6. The Simson Line of D.

The Simson line of $(\alpha'\beta'\gamma')$ is known to be

$$\frac{a^2}{\alpha'^2}\cdot\frac{\alpha}{\beta'\cos C-\gamma'\cos B}+\ldots.=0.$$

So that the Simson line of D is

$$\frac{m-n}{la-mb\cos C-nc\cos B}\cdot\alpha+\ldots.=0.$$

The equation to Simson line which makes angles θ_1, θ_2, θ_3 with the sides of ABC is $a\cot\theta_1.\alpha+\ldots.=0$.

Hence, for the Simson line of D,

$$a\cot\theta_1 \text{ varies as } \frac{m-n}{la-mb\cos C-nc\cos B}.$$

7. To determine the radical axis of the circle on AA' as diameter, and the circle ABC.

Applying the 'power' method to the circle AA',

$$2\rho_2^2=2(BA_1^2-AA_1^2)=2(BA_1^2+AA_1^2)-AA'^2$$
$$=AB^2+A'B^2-AA'^2=2.AB.A'B.\cos B.$$

Hence, $b\rho_2^2=abc\cos B.\frac{n}{n-m}$. So $c\rho_3^2=-abc\cos C.\frac{m}{n-m}$. So that the radical axis is $n\cos B.\beta-m\cos C\gamma=0$. And the circle on AA' is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+\frac{\lambda.abc}{4\triangle^2}(a\alpha+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

Draw AE perpendicular to BC, then E lies on the circle AA'. Its coordinates are $(0,\ b\sin C\cos C,\ c\sin B\cos B)$. Hence the circle AA' is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta-(a\alpha+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

It follows that the radical axis of AA', BB', is

$$\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}-\frac{n\cos A.\alpha-l\operatorname{los} C.\gamma}{n-l}=0,$$

$$\text{or}\quad (m-n)\cos A.\alpha+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

the symmetry of the equation showing that it is the radical axis of all three circles AA', BB', CC'. This radical axis is also the line of the four orthocenters (Art. 3).

8. The radical axis of the diagonal circles is parallel to the Simson Line of D.

If $\lambda\alpha+\mu\beta+\nu\gamma=0$ makes an angle θ with BC, then

$$\cot\theta=\frac{\lambda-\mu\cos C-\nu\cos B}{\nu\sin B-\mu\sin C},$$

so that for the radical axis,

$$a\cot\theta \text{ varies as } \frac{(m-n)}{la-mb\cos C-nc\cos B}.$$

Hence, the radical axis is parallel to the Simson Line (Art. 3).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Note on Problems 267 and 268, by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

In the *Messenger of Mathematics*, 1874, Vol. III, page 137, Mr. Glaisher drew attention to the formula:

$$\tan nx=\frac{n\sin x}{\cos x+}\,\frac{(1^2-n^2)\sin^2 x}{3\cos x+}\,\frac{(2^2-n^2)\sin^2 x}{5\cos x+}\,\frac{(3^2-n^2)\sin^2 x}{7\cos x+\ldots} \qquad (1)$$

x being $<\frac{1}{2}\pi$, and n unrestricted.

For $x=\frac{1}{4}\pi$, $\tan\frac{n\pi}{4}=\frac{n}{1+}\frac{1^2-n^2}{3+}\frac{2^2-n^2}{5+}\frac{3^2-n^2}{7+\ldots}$

Put $n=\frac{1}{2}$, $\tan\frac{\pi}{8}=\frac{1}{2+}\frac{1}{2+}\frac{1}{2+\ldots}$

Put $n=\frac{1}{3}$, $\tan\frac{\pi}{12}=\frac{3}{9+}\frac{6.12}{27+}\frac{15.21}{45+}\frac{24.30}{63+\ldots}$

Put $n=ni$, $\tanh\frac{n\pi}{4}=\frac{n}{1+}\frac{n^2+1}{3+\ldots}$

and as $e^{2x}=-1+\frac{2}{1-\tanh x}$, this gives a continued fraction for $e^{\frac{1}{2}n\pi}$.

On page 65 of Vol. IV, 1875, Mr. Glaisher states that the value for $\tan nx$ was not, as he had thought, original. Vorsselmann and Heer of Utrecht quoted it in 1833 from Euler (*Mém. de l'Acad. de Pétersbourg,* 1813) in the form

$$\tan nx=\frac{n\tan x}{1-}\frac{(n^2-1)\tan^2 x}{3-}\frac{(n^2-2^2)\tan^2 x}{5-}\frac{(n^2-3^2)\tan^2 x}{7-\ldots}. \tag{2}$$

Euler had derived it thus:

$$nz\frac{(1+z)^n+(1-z)^n}{(1+z)^n-(1-z)^n}=1+\frac{(n^2-1)z^2}{3+}\frac{(n^2-2^2)z^2}{3+}\frac{(n^2-3^2)z^2}{5+\ldots}. \tag{3}$$

Vorsellmann derived it from a transformation of

$$\frac{F(\beta+\gamma,\ \beta+1,\ \gamma+1,\ x)}{F(\beta+\gamma,\ \beta,\ \gamma,\ x)}.$$

Glaisher got it from the differential equation corresponding to $y=\cos(n\cos^{-1}x)$ *i. e.*, from $(1-x^2)y_2-xy_1+n^2y=0$ differentiated m times, *i. e.*, $(1-x^2)y_{m+2}-(2m+1)y_{m+1}x+(n^2-m^2)y_m=0$. Then replacing x by $\cos x$, (1) follows at once. Vorsellmann also gave as his own:

$\tan nx=\frac{nt}{1-t^2-}\frac{(n^2-4)t^2}{3(1-t^2)-}\frac{(n^2-16)t^2}{5(1-t^2)-\ldots}$ (4), terminating when n is even,

$=\frac{nt}{1-}\frac{(n^2-1)t^2}{3-t^2-}\frac{(n^2-9)t^2}{5-3t^2-\ldots}$ (5), terminating when n is odd, $2t=\tan x$.

Glaisher points out that (4) is gotten from (2) by substituting $2x$ for x and $\frac{2\tan x}{1-\tan^2 x}$ for $\tan 2x$, and $\frac{1}{2}n$ for n.

In the same way Glaisher gets

$$\tan nx = \frac{nt(1-t^2)}{1-6t^2+t^4-}\,\frac{(n^2-16)t^2\,(1-t^2)^2}{3-18t^2+t^4-}\,\frac{(n^2-64)t^2\,(1-t^2)^2}{5-30t^2+5t^4-\ldots}$$

terminating when n is a multiple of 4.

Putting $n=0$ in (2), (4), (5), we get

$$\tan^{-1}x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+\ldots}, \quad (6) \quad \text{well known.}$$

$$\tan^{-1}x = \frac{x}{1-x^2+}\,\frac{4x^2}{3(1-x^2)+}\,\frac{16x^2}{5(1-x^2)+\ldots} \quad (7)$$

which is (6) with $\frac{2x}{1-x^2}$ for x.

$$\tan^{-1}x = \frac{x}{1+}\,\frac{x^2}{3-x^2+}\,\frac{9x^2}{5-x^2+\ldots}, \quad (8) \quad \text{due to Euler (1779).}$$

Since $\tan nx$ only changes sign if ni be substituted for n and xi for x, (2) gives us

$$\tan nx = \frac{n\tanh x}{1-}\,\frac{(n^2+1)\tanh^2 x}{3-}\,\frac{(n^24)+\tanh^2 x}{5-\ldots}$$

$$\text{or } \tan\left[n \log\sqrt{\frac{1+x}{1-x}}\right] = \frac{nx}{1-}\,\frac{(n^2+1)\,x^2}{3-}\,\frac{(n^2+4)\,x^2}{5-\ldots},$$

a formula also obtainable by replacing n by ni in (3).

Hence, by giving special values to x in (2), (4), and (5), and replacing n by x, we get

$$\tan\frac{\pi x}{4} = \frac{x}{1-}\,\frac{x^2-1}{3-}\,\frac{x^2-4}{5-\ldots} = \frac{x}{1-}\,\frac{x^2-1}{2-}\,\frac{x^2-9}{2-\ldots}$$

$$\tan\frac{\pi x}{3\sqrt{3}} = \frac{x}{1-}\,\frac{x^2-3}{3-}\,\frac{x^2-12}{5-\ldots} = \frac{2x}{3-}\,\frac{4x^2-3}{9-}\,\frac{4x^2-12}{15-\ldots} = \frac{2x}{3-}\,\frac{4x^2-3}{8-}\,\frac{4x^2-27}{12-\ldots}$$

$$\tan\frac{\pi x}{6}=\frac{x}{\sqrt{3}-}\,\frac{x^2-1}{3\sqrt{3}-}\,\frac{x^2-4}{5\sqrt{5}-}$$

For $x=0$, $\pi=\dfrac{3\sqrt{3}}{1+}\,\dfrac{3.1^2}{3+}\,\dfrac{3.2^2}{5+}\,\dfrac{3.3^2}{7+\ldots}=\dfrac{6\sqrt{3}}{3+}\,\dfrac{3.1^2}{9+}\,\dfrac{3.2^2}{15+}\,\dfrac{3.3^2}{21+\ldots}$

$$=\frac{6\sqrt{3}}{3+}\,\frac{3.1^2}{8+}\,\frac{3.3^2}{12+}\,\frac{3.5^2}{16+\ldots}=\frac{6}{\sqrt{3}+}\,\frac{1}{3\sqrt{3}+}\,\frac{2^2}{5\sqrt{3}+}\,\frac{3^2}{7\sqrt{3}+\ldots.}$$

which may also be deduced from (6), (7), and (8).

Glaisher points out the specific advantage of getting a continued fraction from a differential equation, for not only is the result obtained *ab initio* but the remainder at any stage is exhibited in a finite form. He takes $y=\cos(\sin^{-1}\sqrt{z})$ and its differential equation $4(z-z^2)y_2+2(1-2z)y_1+n^2y=0$. Differentiate m times and we have

$$4(z-z^2)y_{m+2}+2(2m+1)(1-2z)y_{m+1}+(n^2-4m^2)y_m=0.$$

Put $z^2=\sin^2\frac{1}{2}x$ and $2n$ for n and we deduce (4).

From $y=\cos\sqrt{x}$ we get $\tan x=\dfrac{x}{1-}\,\dfrac{x^2}{3-\ldots}$

In Vol. VII (1878), page 67, the same author gives further interesting forms:

$$\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}=x-\tfrac{2}{3}x^3+\frac{2.4}{3.5}x^5-\frac{2.4.6}{3.5.7}x^7+\ldots.$$

$$=\frac{x}{1+}\,\frac{1.2x^2}{3-2x^2+}\,\frac{3.4x^2}{5-4x^2+}\,\frac{5.6x^2}{7-6x^2+\ldots}=\frac{1}{x^{-1}+}\,\frac{1.2}{3x^{-1}-2x+\ldots}$$

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}=x+\tfrac{2}{3}x^2+\frac{2.4}{3.5}x^5+\frac{2.4.6}{3.5.7}x^7+\ldots.$$

$$=\frac{x}{1-}\,\frac{1.2x^2}{2x^2+3-}\,\frac{3.4x^2}{4x^2+5-}\,\frac{5.6x^2}{6x^2+7-\ldots}=\frac{1}{x^{-1}-}\,\frac{1.2}{2x+3x^{-1}-\ldots.}$$

Here, putting $x/\sqrt{2}$, we get

$$\frac{2}{\sqrt{3}}\log\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{6}{3+}\,\frac{15}{4+}\,\frac{28}{5+\ldots}$$

1, 6, 15, 28 being the alternate triangular numbers.

$$\frac{\sqrt{2}}{\sqrt{3}}\log\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{1}{\sqrt{2}+}\,\frac{1.2}{2\sqrt{2}+}\,\frac{3.4}{3\sqrt{2}+..}$$

$$\frac{\pi}{2}=\frac{1}{1-}\,\frac{1}{4-}\,\frac{6}{7-}\,\frac{15}{10-}\,\frac{28}{13-...}$$

$$\frac{\pi}{2\sqrt{2}}=\frac{1}{\sqrt{2}-}\,\frac{1.2}{4\sqrt{2}-}\,\frac{3.4}{7\sqrt{2}-}\,\frac{5.6}{10\sqrt{2}-..}$$

For $x=\frac{1}{2}$, $\frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}}{2}=\frac{1}{2+}\,\frac{1.2}{5+}\,\frac{3.4}{8+}\,\frac{5.6}{11+...}$

$$\frac{\pi}{3\sqrt{3}}=\frac{1}{2-}\,\frac{1.2}{7-}\,\frac{3.4}{12-}\,\frac{5.6}{17-...}$$

For $x=1$ in the expansion of $\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}$

$\frac{\log(1+\sqrt{2})}{\sqrt{2}}=\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{3.4}{1+}\,\frac{5.6}{1+...}$, and he compares this with

$$\frac{\pi}{2}-1=\frac{1}{1+}\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+}\,\frac{5.6}{1+...}$$

GEOMETRY.

318. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Is it possible by a straight edge and sect carrier, *i. e.*, without the use of a circle, to construct a mean proportional to two given sects?

Remark by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The value of the length of the mean proportional can be approximately measured without the application of the circle, but it cannot be constructed by pure geometry without such application.

318. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Given three radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

II. Solution by G. W. GREENWOOD, Dunbar, Pa.

Consider first the problem of describing a circle touching two given circles and passing through a given point. Invert with respect to the point; the circles in general invert into circles; draw any common tangent to them

and re-invert into the original circles. The common tangent inverts into a circle through the point and tangent to the given circles. There are, in general, four solutions.

Next, let A, B, C be the centers, and a, b, c the radii, of three given circles. Suppose a is not greater than b or c. Describe circles with radii $b-a$ and $c-a$, and centers B, C, respectively. Let X be the center and x the radius of a circle through A and tangent externally to these two circles. Then a circle, center X and radius $x-a$ will be tangent to the three given circles. By obvious modifications we can, in general, get seven more circles touching the given circles.

We can get the numerical values of the radii and the positions of the centers by means of the linear relations connecting inverse figures.

It would be interesting to consider all the possible cases arising in making these constructions. For example, in the case of the escribed circles of a triangle the sides of the triangle are the limits of three pairs of circles and the nine-point circle is one of the remaining two circles.

CALCULUS.

245. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove or disprove:

$$\int_0^\infty \frac{dx}{\sqrt{(x^2-1)(k^2x^2-1)}}=2F^I(k)+\sqrt{(-1)}.F^I[\sqrt{(1-k^2)}],$$

Legendre's notation, $0<k<1$.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$L=\int_0^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}}=\int_0^1 \frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}}$$

$$+\int_1^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}}=F^I(k)+\int_1^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}}.$$

Let $x=\dfrac{1}{ky}$. Then $\displaystyle\int_0^\infty \frac{dx}{\sqrt{[(x^2-1)(k^2x^2-1)]}}$

$$=\int_0^{1/k} \frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}=F^I(k)+\int_1^{1/k} \frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}.$$

$$\therefore L=2F^I(k)+\int_1^{1/k} \frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}.$$

Let $y=\dfrac{1}{\sqrt{(1-k'^2z^2)}}$, where $k'=\sqrt{(1-k^2)}$.

$$\therefore dy=\frac{k'^2 z dz}{(1-k'^2 z^2)^{\frac{3}{2}}},\quad \frac{1}{\sqrt{[(1-y^2)(1-k^2y^2)]}}=\frac{\sqrt{(-1)}(1-k'^2z^2)}{k'^2 z\sqrt{(1-z^2)}}.$$

$$\therefore \int_1^{1/k}\frac{dy}{\sqrt{[(1-y^2)(1-k^2y^2)]}}=\sqrt{(-1)}\int_0^1\frac{dz}{\sqrt{[(1-z^2)(1-k'^2z^2)]}}$$

$$=\sqrt{(-1)}F^I(k').$$

$$\therefore L=2F^I(k)+\sqrt{(-1)}F^I[\sqrt{(1-k^2)}].$$

Also solved by the Proposer.

246. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

Derive Taylor's Series by the use of the formula for successive integration by parts, and nothing else.

Solution by the PROPOSER.

Assume the identity, $F(a+h)-F(a)=\int_a^{a+h}F'(x)dx$, which on integrating the right hand member by parts,

$$-\Big[(a+h-x)F'(x)\Big]_a^{a+h}+\int_a^{a+h}(a+h-x)F''(x)dx$$

$$=hF'(a)-\Big[\frac{1}{2\,!}(a+h-x)^2F''(x)\Big]_a^{a+h}+\frac{1}{2!}\int_a^{a+h}(a+h-x)^2F'''(x)dx$$

$$=hF'(a)+\frac{h^2}{2\,!}F''(a)+\ldots.+\frac{h^n}{n\,!}F^n(a)+\frac{1}{n!}\int_a^{a+h}(a+h-x)^nF^{n+1}(x)dx.$$

Hence, $F(a+h)-\Big[F(a)+h\,F'(a)+\frac{h^2}{2\,!}F''(a)+\ldots+\frac{h^n}{n\,!}F^n(a)\Big]$

$$(=R)=\frac{1}{n\,!}\int_a^{a+h}(a+h-x)^nF^{n+1}(x)dx.$$

We have assumed here that $F(x)$, $F'(x)$, $F''(x)$, ..., $F^n(x)$, $F^{n+1}(x)$, are all finite and continuous between the limits a and $(a+h)$ of x; now, if we put $A=F^{n+1}(a+\theta h)$, a mean value of $F^{n+1}(x)$ between the limits a and $(a+h)$, we have

$$R=\frac{1}{n!}\int_a^{a+h}(a+h-x)^nAdx=\frac{h^{n+1}}{(n+1)\,!}F^{n+1}(a+\theta h).$$

Also solved by Francis Rust.

MECHANICS.

205. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given two points A and B not in the same horizontal nor in the same vertical line; to find the path from A to B along which a particle will slide from rest under the force of gravity alone so that the average velocity along the curve shall be a maximum.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let the particle start from rest at the point (x_1, y_1), where the axis of x is vertical and the axis of y horizontal. Then

$$v=\frac{ds}{dt}=\sqrt{[2g(x-x_1)]}=\frac{dx}{dt}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Since v=a maximum, t=a minimum.

$$\therefore t=U=\int_{x_1}^{x_2}\frac{\sqrt{[1+(dy/dx)^2]}\,dx}{\sqrt{[2g(x-x_1)]}}=\text{minimum}.$$

By Calculus of Variations, since the expression in the right hand member does not contain y explicitly, the differential of this expression is equal to a constant.

$$\therefore \frac{dy/dx}{\sqrt{[2g(x-x_1)]}\,\sqrt{[1+(dy/dx)^2]}}=C.$$

$$\therefore \left(\frac{dy}{dx}\right)^2=C^2\left[1+\left(\frac{dy}{dx}\right)^2\right][2g(x-x_1)].$$

$$\therefore \frac{dy}{dx}=\pm\frac{C\sqrt{[2g(x-x_1)]}}{\sqrt{[1-2gC^2(x-x_1)]}},$$

the differential equation of a cycloid. This is Bernoulli's famous problem and is solved in almost every work on the Calculus of Variations as well as Dynamics.

206. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A rigid square $ABDC$ made by smooth wires is fixed with A vertically above D. Two small equal spherical elastic beads slide down BD, CD, starting simultaneously from B and C. Find the ratio of their velocities of approach and separation at D, and how far they will separate after impact.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The particles are sliding down inclined planes, inclined at an angle $\beta=\frac{1}{4}\pi$ to the horizon, starting from rest. Let a=side of square, x=distance each particle has moved after a time t. Then the velocity of each is

$$v=\surd(2g\sin\tfrac{1}{4}\pi.x)=\surd(gx\surd2)=\surd(ga\surd2) \text{ at } D.$$

Each particle is moving toward the diagonal at a velocity $=v\sin\frac{1}{4}\pi=\frac{1}{2}v\surd2=\frac{1}{2}\surd[2\surd(2)gx]=\frac{1}{2}\surd[2\surd(2)ga]$ at D. Hence, the particles at D approach each other with a velocity $=2\times\frac{1}{2}\surd[2\surd(2)ga]=\surd[2\surd(2)ga]$.

Let e=the coefficient of restitution. Then, since the particles impinge at right angles, they will separate with a velocity $v_1=e\surd[2\surd(2)ga]$, and will ascend DB, DA, respectively, with a velocity $v_2=e\surd(ga\surd2)$. Each will ascend a distance $=v_2^2/2g\sin\frac{1}{4}\pi=v_2^2/g\surd2=e^2a$.

Therefore, they separate, after impact, a distance $=2e^2a\sin\frac{1}{4}\pi=e^2a\surd2$.

If $e=1$, they return to their starting points.

AVERAGE AND PROBABILITY.

190. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line is drawn at random across a regular $2n$-gon; what is the chance that it crosses parallel sides?

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Let AB, CD be two parallel sides. Through the center of the polygon O draw PQ to represent the direction of the random line.

Let $AO=r$, $\angle AOB=\pi/n=\beta$, $\angle AOP=\theta$.

For all possible positions PQ may be regarded as lying in one quadrant. The actual line parallel to PQ may hold any position within breadth of plane HF (H being the most remote vertex from QP, and F foot of perpendicular from H on QP) for all positions, and AE (A being the nearest vertex to QP and E foot of perpendicular from A to QP) for all favorable positions.

$\therefore$ The chance is $p=AE/HF$; $AE=r\sin\theta$, $HF=r\cos\theta$ for n even; $HF=r\cos(\frac{1}{2}\beta-\theta)$ for n odd. For n even,

$$p=\frac{2n}{\pi}\int_0^{\frac{1}{2}\beta}\tan\theta\, d\theta=\frac{2n}{\pi}\log(\sec\tfrac{1}{2}\beta).\quad \therefore p=\frac{2n}{\pi}\log\left(\sec\frac{\pi}{2n}\right).$$

For n odd, $$p=\frac{2n}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\theta}{\cos(\frac{1}{2}\beta-\theta)}d\theta=\frac{n}{\pi}(\beta\sin\tfrac{1}{2}\beta+2\cos\tfrac{1}{2}\beta\log\cos\tfrac{1}{2}\beta).$$

$$\therefore p=\frac{n}{\pi}\left[\frac{\pi}{n}\sin\frac{\pi}{2n}+2\cos\frac{\pi}{2n}\log\left(\cos\frac{\pi}{2n}\right)\right].$$

191. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Two random lines cut a given circle. What is the chance that they intersect within the circle?

Solution by HENRY HEATON, Belfield, N. D., and the PROPOSER.

Let x=the distance of one of the lines from the center of the circle, and θ=the angle between the lines. The length of the part of the first line lying within the circle is $2\sqrt{(a^2-x^2)}$. For given values of x and θ the chance of intersection is

$$\frac{2\sin\theta\sqrt{(a^2-x^2)}}{2a}=\frac{\sin\theta}{a}\sqrt{(a^2-x^2)},$$

and the required chance is

$$P=\int_0^a\int_0^{\frac{1}{2}\pi}\frac{\sin\theta}{a}\sqrt{(a^2-x^2)}\,dx\,d\theta\div\int_0^a\int_0^{\frac{1}{2}\pi}dx\,d\theta=\frac{2}{a\pi}\int_0^a\sqrt{(a^2-x^2)}\,dx=\tfrac{1}{2}.$$

Also solved by G. B. M. Zerr, who gets ⅛ for the result. His solution will be published in the next issue of the MONTHLY.

PROBLEMS FOR SOLUTION.

ALGEBRA.

293. Proposed by C. E. WHITE, Vanderbilt University, Nashville, Tenn.

Prove by mathematical induction that $\frac{(x-a)^{m-1}}{(m-1)!}f^{m-1}(a)+\frac{(x-a)^{m-2}}{(m-2)!}+\dots\frac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a)$ will be the remainder when $f(x)$ is divided by $(x-a)^m$.

294. Proposed by O. L. CALLECOT, Gettysburg, S. Dak.

Find the limit of $\sum_{n=1}^{n=\infty}\frac{2(n^2+3n+3)}{n(n+1)(n+2)(n+3)}$.

GEOMETRY.

326. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

The circle C of radius pR encloses the circles A_1B_1 of radii R and $(p-1)R$, respectively; the circle B_1 is tangent to $A_1B_1C_1$; the circle B_2 is tangent to AB_1C; the circle B_3 to AB_2C, ..., B_n to $AB_{n-1}C$. Find the radius of the circle B_n.

327. **Proposed by J. C. CORBIN, Pine Bluff, Ark.**

In triangle ABC, the triangle DEF is formed by joining the feet of the medians and four parallelograms are also formed, viz., $AEDF$, $BFED$, and $CEFD$. Let a, b, c, d, e, f represent the three medians of ABC, and the three sides of DEF. Then the sum of the squares of the six diagonals equals the sum of the squares of the twelve sides of the parallelograms, which are equal in sets of four. That is, $a^2+b^2+c^2+d^2+e^2+f^2=4(d^2+e^2+f^2)$, or $a^2+b^2+c^2=3(d^2+e^2+f^2)=3/4(AB^2+BC^2+CA^2)$.

328. **Proposed by CHARLES GILPIN, JR., Philadelphia, Pa.**

A sphere with the radius R is divided into two segments by a plane passed through it half way between the center and circumference. The smaller segment is divided into two parts by a plane passed through it at right angles to the base and cutting it half way between its center and circumference. Required the contents of the two parts of the segment.

CALCULUS.

250. **Proposed by V. M. SPUNAR, East Pittsburg, Pa.**

Differentiate $(\log^n x)$.

251. **Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.**

Find in terms of x the functions c_1x and c_2x defined, respectively, by the relations (a) $x\log(c_1x)=c_1x\log x$,
(b) $x\log x=c_2x\log(c_2x)$.

MECHANICS.

210. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

A rigid triangle is formed of three weightless, smoothly jointed, rigid rods BC, CA, AB. At their mid points D, E, F, respectively, are small, smooth rings, through which passes an endless, stretched, elastic string, forming the triangle DEF. Find by graphical construction the reaction at the joints.

211. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

A smooth elliptic wire, axis vertical, has a small ring sliding on it, connected by elastic strings with each focus. Either string is just unstretched when the ring is nearest the corresponding focus. The modulus of elasticity is W/n, where Woz. is the weight of the ring. Find the distance of the ring from the upper focus in the different positions of equilibrium, and in each case discuss the nature of the equilibrium.

MISCELLANEOUS.

175. **Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.**

If x and z are connected by the relation $z=zf(x)+x\phi(z)$, find the value of $f(z)$ in the form of a power series in x with constant coefficients. In particular, give such a value of z when $z=z\sin x+x\cos z$.